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Mathematical Reviews

Edited by

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J. V. Wehausen, *Executive Editor*

Vol. 15, No. 11 December, 1954 pp. 923–1012

The index issue for Vol. 15 will appear as Part 2 of one of the issues of Vol. 16.

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MATHEMATICAL REVIEWS

Published monthly, except August, by

THE AMERICAN MATHEMATICAL SOCIETY, Prince and Lemon Streets, Lancaster, Pennsylvania

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INDIAN MATHEMATICAL SOCIETY
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Editorial Office

MATHEMATICAL REVIEWS, 80 Waterman St., Providence 6, R. I.

Subscriptions: Price $20 per year ($10 per year to members of sponsoring societies). Checks should be made payable to MATHEMATICAL REVIEWS. Subscriptions may be addressed to MATHEMATICAL REVIEWS, Lancaster, Pennsylvania, but should preferably be addressed to the American Mathematical Society, 80 Waterman St., Providence 6, R. I.

This publication was made possible in part by funds granted by the Carnegie Corporation of New York, the Rockefeller Foundation, and the American Philosophical Society held at Philadelphia for Promoting Useful Knowledge. Its preparation is also supported currently under a contract with the Office of Scientific Research, Headquarters, Air Research and Development Command, U. S. Air Force. These organizations are not, however, the authors, owners, publishers, or proprietors of this publication, and are not to be understood as approving by virtue of their grants any of the statements made or views expressed therein.

Entered as second-class matter February 3, 1940 at the post office at Lancaster, Pennsylvania, under the act of March 3, 1879. Accepted for mailing at special rate of postage provided for in paragraph (d-2), section 34.40, P. L. and R. of 1948, authorized November 9, 1940.

Mathematical Reviews

Vol. 15, No. 11 DECEMBER, 1954 Pages 923–1012

HISTORY

Venkachalam Iyer, R. The Hindu abacus. Scripta Math. 20, 58–63 (1954).

Aaboe, Asger. al-Kāshī's iteration method for the determination of sin 1°. Scripta Math. 20, 24–29 (1954).

van der Waerden, Bartel L. Bemerkungen zu den Handlichen Tafeln des Ptolemaios. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1953, 261–272 (1954).

Dijksterhuis, E. J. Die Integrationsmethoden von Archimedes. Nordisk Mat. Tidskr. 2, 5–23 (1954).

Tenca, Luigi. Lettere di scienziati dello Studio Padovano del principio del 1700. Ist. Veneto Sci. Lett. Arti. Atti Cl. Sci. Mat. Nat. 111, 83–102 (1953).

Hofmann, Josef Ehrenfried. Die Mathematik an den altbayerischen Hochschulen. Abh. Bayer. Akad. Wiss. Math.-Nat. Kl. (N.F.) no. 62, 26 pp. (1954).

de Dainville, François. L'enseignement des mathématiques dans les Collèges Jésuites de France du XVI^e au XVIII^e siècle. II. Rev. Hist. Sci. Appl. 7, 109–123 (1954).

Rédei, László. Life and work of Mihály Bauer. Mat. Lapok 4, 241–262 (1953). (Hungarian)
A list of Bauer's mathematical papers is included.

Villa, Mario. Commemorazione: Luigi Berzolari. Rend. Sess. Accad. Sci. Ist. Bologna 57 (1952–1953), 27–41 (1 plate) (1954).
A list of Berzolari's mathematical papers is included.

Sz.-Nagy, Gyula. Dreiteilung des Winkels nach Johann Bolyai. Mat. Lapok 4, 84–86 (1953). (Hungarian. Russian and German summaries)

Onoranze alla memoria di Guido Castelnuovo. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 13, 1–49 (1 plate) (1954).
A number of short addresses by E. Bompiani, F. Severi, G. U. Papi, S. Visco, B. Segre, and F. P. Cantelli and a list of Castelnuovo's published work. Castelnuovo was born 14 Aug. 1865 and died 12 Dec. 1953.

Segre, Beniamino. Obituary: Fabio Conforto. Archimede 6, 91–94 (2 plates) (1954).

In memoria di Fabio Conforto. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) 13, 199–218 (1954).
A short biography followed by addresses by G. U. Papi, S. Visco, F. Severi, and E. Bompiani. Conforto was born on 13 Aug. 1909 and died 24 Feb. 1954.

Elenco delle pubblicazioni del prof. Fabio Conforto. Archimede 6, 127–130 (1954).

Court, Nathan Altshiller. Desargues and his strange theorem. Scripta Math. 20, 5–13 (1954).

Terracini, Alessandro. Gino Fano, 1871–1952. Cenni commemorativi. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 87, 350–360 (1953).

***Fedorov, E. S. Načala učeniya o figurah. [Elements of the study of figures.]** Izdat. Akad. Nauk SSSR, Moscow, 1953. 410 pp. (5 plates). 16.35 rubles.
Fedorov's work appeared originally in Zapiski Mineralogičeskih Obščestva (2) 21, 1–279 (1885). This edition contains also an essay on the significance of this work for crystallography by V. A. Frank-Kameneckiĭ and notes on the work by O. M. Anšeles, I. I. Šafranovskiĭ and Frank-Kameneckiĭ.

Tietze, Heinrich. Nachruf: Gustav Herglotz. Bayer. Akad. Wiss. Jbuch. 1953, 188–194 (1 plate) (1954).
Herglotz was born 2 Feb. 1881 and died 22 March 1953.

Tietze, Heinrich. Aus Gesprächen mit Gustav Herglotz. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1953, 163–167 (1954).

Sergescu, P. Notice nécrologique: Pierre Humbert. Arch. Internat. Hist. Sci. (N.S.) 7, 181–183 (1954).

Blaschke, Wilhelm. Keplero e Galileo. Giorn. Mat. Battaglini (5) 2(82), 309–334 (1954).

***Geronimus, J. L. Alexej Nikolajewitsch Krylow (1863 bis 1945). Näherungsrechnungen in der Schwingungs- und Elastizitätstheorie.** Verlag Technik, Berlin, 1953. 56 pp.
Translation of an essay from the author's "Essays on the work of the leaders of Russian mechanics" [Moscow, 1952; these Rev. 15, 275].

Hyers, D. H. Aristotle D. Michal, 1899–1953. Math. Mag. 27, 237–244 (1954).
A list of Michal's published mathematical work is included.

Frank, Philipp. The work of Richard von Mises: 1883–1953. Science 119, 823–824 (1954).

Sauer, R. Nachruf: Richard von Mises. Bayer. Akad. Wiss. Jbuch. 1953, 194–197 (1 plate) (1954).
Von Mises was born 19 April 1883 and died 14 July 1953.

***Andrade, E. N. da C. Sir Isaac Newton.** Collins, London, 1954. 140 pp. (1 plate) $1.05.

Natucci, Alpinolo. Nel primo centenario della nascita di Salvatore Pincherle. Giorn. Mat. Battaglini (5) **2**(82), 335–342 (1954).

***Pincherle, Salvatore. Opere Scelte. Vol. I.** A cura della Unione Matematica Italiana. Edizioni Cremonese, Roma, 1954. ii+397 pp. (1 plate). 3500 Lire.

A selection of 38 of Pincherle's mathematical papers are to be published in two volumes. This volume contains Amaldi's commemorative address (1937), a complete list of his publications, Pincherle's own discussion of his work [Acta Math. **46**, 341–362 (1925)] and fourteen other papers.

Hadamard, Jacques. Le centenaire de Henri Poincaré. Rev. Hist. Sci. Appl. **7**, 101–108 (1954).

Hofmann, Jos. E. Über Portas Quadratur krummlinig begrenzter ebener Figuren. Arch. Internat. Hist. Sci. (N.S.) **6**, 193–208 (1953).

This is an interesting account of a sixteenth century geometrician Della Porta (1538–1615) whose books have received but little attention: the Magia naturalis [Naples, 1558] and the Elementa curvilineorum [Naples, 1601; Rome, 1610]. Porta's work is a study, by Euclidean methods, of areas bounded by circular arcs and straight lines; the chief proposition is the proof that the sum of the two lunes of any right-angled triangle is equal to the area of the triangle, but it is followed by a false inference, that each lune separately can be squared. Hofmann has given a sympathetic description of a mathematician who, while not among the creative artists, yet wrought with care and finish; and the essay throws light upon the geometry of that era. *H. W. Turnbull* (Millom).

Hofmann, Jos. E. Über das unmittelbare Nachwirken der Portaschen Quadratur krummlinig begrenzter ebener Figuren. Arch. Internat. Hist. Sci. (N.S.) **7**, 16–34 (1954).

This is a sequel to the paper reviewed above and deals with the squaring of a lune of a right-angled triangle following the methods of Porta. An account is given of work by three investigators: Paolo Aurineto (early 17th century), Artus de Lionne (1583–1663) and Vincent Léotaud [1595–1672). Enough detail is included to shew the skill of these forgotten geometers in devising rectilinear approximations (quadrilateral and pentagonal) for the curvilinear area: the triangles with sides $1:1:\sqrt{2}$ and $3:4:5$ receive special treatment. The assertion by both Lionne and Léotaud that the area of each lune (for the general right-angled triangle) is transcendental, is taken seriously. A useful list of names mentioned in the text and in the many footnotes is added. *H. W. Turnbull* (Millom).

***Demtchenko, Basile. Les idées, les travaux, les écrits de M. Dimitri Riabouchinsky.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. i–xi. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

***Demtchenko, Basile. The ideas, work and writings of Dimitri Riabouchinsky.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. xiii–xxii. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

Translation of the paper listed above.

***Liste chronologique des publications de Dimitri P. Riabouchinsky.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. xxiii–xxxv. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

***Geronimus, J. L. Ossip Iwanowitsch Somow (1815 bis 1876). Einige Probleme der höheren Dynamik.** Verlag Technik, Berlin, 1954. 55 pp. DM 3.60.

Translation of an essay from the author's "Essays on the life and work of the leaders of Russian mechanics" [Moscow, 1952; these Rev. **15**, 275].

Obituary: Gyula Szökefalvi-Nagy. Mat. Lapok **4**, 81–83 (1953). (Hungarian)

***Timoshenko, Stephen P. The collected papers of Stephen P. Timoshenko.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. xxv+642 pp. (1 plate). $12.20.

The early papers in Russian are listed only by title. There is also a short biographical sketch by D. H. Young.

FOUNDATIONS

Fitch, Frederic B. A simplification of basic logic. J. Symbolic Logic **18** (1953), 317–325 (1954).

The formulation of Fitch's basic logic K [same J. **7**, 105–114 (1942); these Rev. **4**, 125] is simplified in the following respect. The proper ancestral of a relation is now shown to be definable in terms of the other concepts of the system. In order to achieve the effect of the two rules needed for the proper ancestral, a rule of triple abstraction is employed, but no rules of n-tuple abstraction ($n>3$). Let K^* be the system one gets from K by adding rules of n-tuple abstraction for every $n>2$. Using methods previously developed [ibid. **9**, 57–62 (1944); these Rev. **6**, 197], Fitch shows that K^* is a minimum relationally basic calculus in an appropriate sense, whereas K is merely monadically, dyadically, and triadically basic but not minimum. Another system $K^\dagger$ is formulated which is also relationally basic but not minimum. *R. M. Martin* (Philadelphia, Pa.).

Myhill, John. Arithmetic with creative definitions by induction. J. Symbolic Logic **18**, 115–118 (1953).

This paper presents a strong kind of arithmetic, the axioms of which consist of Peano's Postulates, statements characterizing the Hilbert μ-function (the smallest number such that), and a strong rule of recursive definition. It is shown that the ordinary recursion equations for '+' and '×' are forthcoming, that the consistency of ordinary number theory can be proved within this system, and that the system allows the inexhaustible use of 'creative' definitions in the sense of Leśniewski. *R. M. Martin.*

Quine, W. V. Interpretations of sets of conditions. J. Symbolic Logic **19**, 97–102 (1954).

Concise account of the construction of arithmetic models for consistent sets of formulae of the lower predicate calculus. *G. Kreisel* (Reading).

McNaughton, Robert. A non-standard truth definition. Proc. Amer. Math. Soc. **5**, 505–509 (1954).

Z_C is an extension of the Zermelo-Fraenkel set theory, with an impredicative axiom schema; Z_C' is a subsystem of Z_C which has the same formulae as Z_C, but only a finite number of axioms. Wang [Math. Ann. **125**, 56–66 (1952); these Rev. **14**, 439] observed that an inner model for Z_C' could be defined in Z_C such that the sets of the model constitute a set w which is denumerable in Z_C.

The author uses this result to obtain a truth definition Tr' for Z_C' by carrying out in Z_C the following construction: in any closed formula A (Gödel number a) the quantifiers are relativized to range over w, while the predicates remain absolute; if the formula A_w (of Z_C) is this relativized version of A then $\mathrm{Tr}'(a)$ means: A_w is true, or: A is true in Wang's model. Tr' is called a 'truth definition' because, for each A, $\mathrm{Tr}'(a) \leftrightarrow A_w$ can be proved (in Z_C), and 'non-standard' because, in general, $\mathrm{Tr}'(a) \leftrightarrow A$ cannot be proved: this last fact explains why Tr' can be defined by means of the symbolism of Z_C' itself.

Reviewer's notes: On p. 507, line 6 from bottom, the author repeats the familiar assertion that a standard truth definition requires a higher type of variable; this is false: Hilbert and Bernays [Grundlagen der Mathematik, Bd. II, Springer, Berlin, 1939, p. 339] provided a standard truth definition for the system Z of arithmetic in a system obtained from Z by adding (i) to its predicate symbols $a+b=c$, $a \cdot b=c$ the new symbol $M(n, k)$, and (ii) to its axioms an axiom for $M(n, k)$, namely, lines 20–22 of p. 339. Actually, if (S) is any system containing arithmetic and if the consistency of a set theory (S') can be proved in (S), then a non-standard truth definition for (S') can be constructed in (S) from the classical Skolem model [cf., e.g., remark 2 on p. 47 of the reviewer's note in Proc. XIth international congress of philosophy, 1953, v. XIV, North-Holland Publ. Co., Amsterdam, 1953, pp. 39–49; these Rev. 15, 668]. *G. Kreisel* (Reading).

Yablonskiĭ, S. V. On functional completeness in a three-valued calculus. Doklady Akad. Nauk SSSR (N.S.) **95**, 1153–1155 (1954). (Russian)

The author states without proof a theorem to the effect that in order for a set of functions in a three-valued calculus to be functionally complete it is necessary and sufficient that the set contain at least one function not belonging to each of 18 specified classes of functions; consequently from any such complete set one can select a complete subset containing at most 18 functions. This is compared to a similar result of Kuznecov, for which the proof is also unpublished.

H. B. Curry (State College, ~~Md.~~ Pa.).

Rose, Alan. Sur les fonctions définissables dans une logique à un nombre infini de valeurs. C. R. Acad. Sci. Paris **238**, 1462–1463 (1954).

Extension of McNaughton's Theorem 2 in J. Symbolic Logic **16**, 1–13 (1951) [these Rev. **13**, 3] to an infinitely-many-valued logic. *G. Kreisel* (Reading).

Janiczak, A. Some remarks on partially recursive functions. Colloquium Math. **3**, 37–38 (1954).

E. L. Post [Bull. Amer. Math. Soc. **50**, 284–316 (1944), p. 310; these Rev. **6**, 29] has constructed a recursively enumerable set with infinite complement which intersects all infinite recursively enumerable sets. By the use of partially recursive functions the author obtains a simpler example of such a set. *G. Kreisel* (Reading).

Janiczak, A. On the reducibility of decision problems. Colloquium Math. **3**, 33–36 (1954).

Independent proof of a weaker form of Theorem XI, p. 293, of Kleene's "Introduction to metamathematics" [Van Nostrand, New York, 1952; these Rev. **14**, 525]. *G. Kreisel* (Reading).

Robinson, Abraham. On predicates in algebraically closed fields. J. Symbolic Logic **19**, 103–114 (1954).

Während sich die algebraische Geometrie auf die Untersuchungen von Mannigfaltigkeiten, d.h. der Mengen von Punkten $\xi_1, \cdots, \xi_n$, die ein System algebraischer Gleichungen erfüllen, beschränkt, gestattet die "Metamathematik" (die trotz ihres Namens echte Mathematik ist) die Ausdehnung der Untersuchung auf alle Mengen V_Q von Punkten, die eine "elementare" Aussagenform $Q(x_1, \cdots, x_n)$ erfüllen. Q ist dabei aus algebraischen Gleichungen mit Hilfe der logischen Verknüpfungen (einschl. Quantoren) zusammengesetzt. Ist F ein (kommutativer) Körper, dann gibt es in $F(x_1, \cdots, x_n)$ eine Kette von Idealen

$$J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{2k+1} \quad (k \geqq 0),$$

so dass in jedem algebraisch abgeschlossenem Oberkörper F^* von F gilt: $V_Q = (V_0 - V_1) \cup (V_2 - V_3) \cup \cdots \cup (V_{2k} - V_{2k+1})$, wenn V_i die zu J_i gehörige Mannigfaltigkeit über F^* ist. Wie Verf. bemerkt, liefert auch die Methode von Tarski und McKinsey [A decision method for elementary algebra and geometry, 2d ed., Berkeley, Calif., 1951; diese Rev. **13**, 423] einen Beweis dieses Satzes. Der vorliegende Beweis verdient jedoch eigenes Interesse da er von der Eliminationstheorie keinen Gebrauch macht.

Als wesentliches Hilfsmittel wird zunächst aus dem Gödelschen Vollständigkeitssatz gefolgert, dass eine elementare Aussage, die in einem algebraisch abgeschlossenen Oberkörper von F gilt, in allen solchen Oberkörpern gilt. Der Beweis wird dann unter Heranziehung idealtheoretischer Sätze (Hilbert-Netto) durch eine Induktion—auf Grund der Maximalbedingung für Ideale in Polynomringen —geführt. *P. Lorenzen* (Bonn).

Sobociński, Bolesław. Note on a modal system of Feys-von Wright. J. Computing Systems **1**, 171–178 (1953).

It is shown in this paper that Feys' System T [Revue Philos. Louvain **48**, 478–509 (1950)] is equivalent with the System M of von Wright [An essay in modal logic, North-Holland Publ. Co., Amsterdam, 1951; these Rev. **13**, 614]. It is also shown that these systems contain infinitely many modalities. *R. M. Martin* (Philadelphia, Pa.).

Brouwer, L. E. J. Further addenda and corrigenda on the role of the principium tertii exclusi in mathematics. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 109–111 (1954). (Dutch)

The example of a continuous monotone real full function which is nowhere derivable, given by the author [J. Reine Angew. Math. **154**, 1–7 (1924)], is incorrect; on closer inspection it reveals itself as almost everywhere derivable. Here another function is constructed which truly has the stated properties. *A. Heyting* (Amsterdam).

ALGEBRA

Sprott, D. A. A note on balanced incomplete block designs. Canadian J. Math. 6, 341–346 (1954).

By means of R. C. Bose's first module theorem [Ann. Eugenics 9, 353–399 (1939); these Rev. 1, 199] the author constructs balanced incomplete block designs with v a prime power, $b=mv$ and $v=mk+1$, $k=\lambda+1$; $v=2m(2\lambda+1)$, $k=2\lambda+1$; $v=2m(2\lambda-1)$, $k=2\lambda$ and under certain conditions also $v=4m(4\lambda+1)$, $k=4\lambda+1$. *H. B. Mann.*

Brenner, J. L. Bounds for determinants. Proc. Nat. Acad. Sci. U. S. A. 40, 452–454 (1954).

Let $A=(a_{ij})$ be an $n\times n$ matrix and let the n rows of A be divided into s disjoint groups of $r_1, r_2, \cdots, r_s$ rows respectively, where $n=r_1+r_2+\cdots+r_s$. It is shown that $\det A\neq 0$ if in each group of rows the absolute value of the principal minor on these rows exceeds the sum of the absolute values of all the non-principal minors on the same rows. This is a substantial generalization of the well-known theorem on determinants with dominant principal diagonal, to which the new theorem reduces when $r_1=r_2=\cdots=r_n=1$. The paper also contains explicit lower and upper bounds for $\det A$ in terms of the absolute values of the minors of A. *W. Ledermann* (Manchester).

Markovitch, D. Quelques propriétés d'une matrice carrée spéciale. Bull. Soc. Math. Phys. Serbie 5, no. 3–4, 45–51 (1953). (Serbo-Croatian summary)

The matrix considered is the $n\times n$ matrix

$$A=\begin{pmatrix} a_0, a_1, \cdots, a_{n-1} \\ I_{n-1}, O \end{pmatrix}$$

where I_{n-1} is the $(n-1)\times(n-1)$ unit matrix, and O is a column of $n-1$ zeros. Polynomial recursions for the elements of A^n are found. The limit $(1/a_0^{(n)})/A^n$ is computed. *J. L. Brenner* (Aberdeen, Md.).

Wong, Y. K. On non-negative-valued matrices. Proc. Nat. Acad. Sci. U. S. A. 40, 121–124 (1954).

Let $A=(a_{ij})$ be an $n\times n$ matrix with non-negative elements, and let $s_j^{(k)}$ be the sum of the first k elements of the jth column of A. Setting $s_j=s_j^{(n)}$, the author defines $\|A\|=\max(s_1, \cdots, s_n)$, and assumes throughout that $\|A\|\leq 1$. Under this hypothesis, the following properties of A are shown to be logically equivalent: (i) $I-A$ is non-singular; (ii) $A\sim B$, where B satisfies $s_k^{(k)}<1$ for $k=1, \cdots, n$; (iii) there exist $k<n$ and $e>0$ such that

$$\|(s_1, \cdots, s_n)A^k\|\leq e<1;$$

(iv) $\lim_q \|A^q\|^{1/q}<1$; (v) $A\sim B$, where B satisfies

$$\sum_{i=1}^{k-1} s_i^{(k)}a_{ik}<1-a_{kk}$$

for $k=2, \cdots, n$, and $a_{11}<1$. (Here, $A\sim B$ means that B is obtainable from A by permutations of the same rows and columns.) As a corollary to this theorem, it is proved that if A is not nilpotent, and if A has a simple characteristic root λ_1 with greatest modulus, then

$$0<\lambda_1=\lim_q \|A^q\|^{1/q}=e^b, \quad b=\inf_p [p^{-1}\log\|A^p\|]>-\infty.$$

Various applications of these results are made, including some estimates for $\beta'(A+A^2+\cdots)$, where β' is a row vector with non-negative coordinates. *I. Reiner.*

Jarosch, Wilhelm. Matrizenbänder. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 159, 287–296 (1950).

Starting with a given scale of relation

$$a_{n+r}+c_1a_{n+r-1}+\cdots+c_na_r=0 \quad (c_n\neq 0),$$

write on successive lines n recurring sequences $\{a_1^{(0)}, a_2^{(0)}, \cdots\}$ $(i=1, \cdots, n)$ having this scale, to form what the author calls a strip of matrices (Matrizenband). From this strip a sequence of matrices $A_0, A_1, \cdots$ is obtained by letting A_k consist of the $(k+1)$th, $\cdots$, $(k+n)$th columns of the strip. If C denotes the transpose of the companion matrix of $x^n+c_1x^{n-1}+\cdots+c_n=0$, then $A_r=A_0C^r$. The special cases where $A_0=E$, and also where A_k is of the form $(a_{ij}^{(k)})$, with $a_{ij}^{(k)}=x_{k+i+j-2}$ $(i, j=1, \cdots, n)$, are considered in detail. A multiplication is defined for certain types of strips, and is then interpreted in terms of operations on the generating functions of the corresponding recurring sequences. There are a few minor misprints in the paper. *I. Reiner.*

Foulkes, H. O. Plethysm of S-functions. Philos. Trans. Roy. Soc. London. Ser. A. 246, 555–591 (1954).

The study of the coefficients $c_{\lambda\mu\nu}$, where

$$\{\lambda\}\otimes\{\mu\}=\sum c_{\lambda\mu\nu}\{\nu\}$$

and $\{\lambda\}$ is the Schur function related to the partition (λ), can also be made with reference to the equation

$$[\lambda]\odot[\mu]=\sum c_{\lambda\mu\nu}[\nu],$$

where $[\lambda]$ is an irreducible representation of the symmetric group. In this form the operation $\odot$, and so $\otimes$, can be explicitly defined but so far no explicit formula, not involving characters [cf. Robinson, Canadian J. Math. 2, 334–343 (1950); these Rev. 12, 74] has been given for $c_{\lambda\mu\nu}$ in the general case. The formulation of invariant theory in terms of these general ideas is of real interest but has led to no great practical advance, largely because of the difficulties involved in the above reduction. The present paper is devoted to a systematic study of the problem and numerous properties of the c's are proved or conjectured in certain special cases. For example, a practical method is given for obtaining the reduction of $\{m\}\otimes\{\mu\}$ where m is integral and (μ) is a partition of 4. Thence formulae for the c's are obtained if $\{\lambda\}$ is of the form:

$$\{4m-k, k\}, \quad \{m+k, m+k, m-k, m-k\},$$
$$\{m+k, m, m, m-k\}\,\{4m-2k, k, k\},$$

where $m\geq k\geq 0$. Incidentally, formulae are obtained for the coefficient of $\{\lambda\}$ in $\{m\}^n$. The paper closes with five main theorems of a more general character, of which the following is a sample. "The coefficient of $\{\lambda_1+1, \lambda_2+1, \cdots, \lambda_n+1\}$ in $\{m+1\}\otimes\{\mu\}$ is equal to the coefficient of $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ in $\{m\}\otimes\{\bar{\mu}\}$ where (μ) is a partition of n and $(\bar{\mu})$ is conjugate to (μ)." *G. de B. Robinson* (Toronto, Ont.).

Abstract Algebra

Sorkin, Yu. I. On the imbedding of latticoids in lattices. Doklady Akad. Nauk SSSR (N.S.) 95, 931–934 (1954). (Russian)

For a given class $\mathfrak{A}$ of algebras, T. Evans [J. London Math. Soc. 26, 64–71 (1951); 28, 76–80 (1953); these Rev. 12, 475; 14, 839] analyzed the problem of embedding in-

complete $\mathfrak{A}$-algebras (i.e., operations not necessarily always defined) in $\mathfrak{A}$-algebras. One of his examples was the embedding of partially ordered sets in lattices. The author points out that the usual postulates for a partially ordered set are not in the form of an incomplete lattice (=latticoid) in the above sense. He lays down axioms in the desired form, and finds that there is a unique minimal complete set of such axioms, in contrast to his results for lattices [Ukrain. Mat. Žurnal 3, 85–97 (1951); these Rev. 14, 612]. The proof of embeddability is formulated accordingly. [This appears closely related to work of R. P. Dilworth, Trans. Amer. Math. Soc. 57, 123–154 (1945); these Rev. 7, 1.] Then he shows explicitly that any countable latticoid can be embedded in a latticoid with 3 generators. This contains as a special case the result of the reviewer for free lattices [Ann. of Math. (2) 43, 104–115 (1942); these Rev. 3, 261].

P. M. Whitman (Silver Spring, Md.).

Riguet, Jacques. Sur l'extension du calcul des relations binaires au calcul des matrices à éléments dans une algèbre de Boole complète. C. R. Acad. Sci. Paris 238, 2382–2385 (1954).

Given a set E and a complete distributive lattice L, the author considers vectors and matrices as mappings of E into L, and $E\times E$ into L, respectively, with obvious extensions of the operations and partial order of L. For vectors u, v, and matrices α, β, define

$$(u|v)=\mathsf{V}_{z\in E}[u(z)\wedge v(z)];\quad (u\otimes v)(x)=u(x)\wedge v(x);$$
$$\alpha u(x)=\mathsf{V}_{z\in E}[\alpha(z,x)\wedge u(z)];$$
$$\beta\alpha(x,y)=\mathsf{V}_{z\in E}[\beta(z,y)\wedge\alpha(x,z)];$$
$$\delta(x,y)=1 \text{ or } 0 \text{ as } x=y \text{ or } x\neq y;\quad \alpha'(x,y)=\alpha(y,x),$$

for $x, y\in E$ and α, β matrices. Then analogues of formulas about relations [Riguet, Bull. Soc. Math. France 76, 114–155 (1948); these Rev. 10, 502] hold. A matrix is disjunctive in the sense of L. Löwenheim [Math. Ann. 68, 169–207 (1910); 73, 245–272 (1913); 79, 223–236 (1918)] if and only if $\alpha\alpha'\leqq\delta\leqq\alpha'\alpha$; if u and v are disjunctive systems then $v\leqq\alpha u$, $u\leqq\alpha' v$, and $u\otimes v\leqq\alpha$ are equivalent; the analogous statement in the theory of relations is that $y\in R(x)$, $x\in R^{-1}(y)$, and $(x,y)\in R$ are equivalent. Other results of Löwenheim and R. D. Luce [Proc. Amer. Math. Soc. 3, 382–388 (1952); these Rev. 14, 347] are then easily found.

P. M. Whitman (Silver Spring, Md.).

Schützenberger, Marcel Paul. Sur une définition combinatoire des espaces vectoriels classiques. C. R. Acad. Sci. Paris 238, 2487–2488 (1954).

Let E be a set of "points" and α be a symmetric binary relation on E. If, for $P\subset E$, $P^*=\{a; a\,\alpha\, b \text{ for every } b\in P\}$, then the correspondence $P\to P^{**}$ defines a closure operation on the lattice of all subsets of E. For subsets P and Q of E, define $P+Q=(P\cup Q)^{**}$. Such a structure $\mathcal{E}=(E,\alpha)$ is called a "classical bilinear relation" (CBR) if and only if for $a,b,c\in E$, $a\neq b$, $a\,\alpha\, b$, $c\not\subset a^*\cap b^*$ imply $c^*\cap(a+b)$ is a unique point. The author states that the classical hermitian, orthogonal, and symplectic vector spaces are CBR.

R. E. Johnson (Northampton, Mass.).

Rokos, P. Generalisation of theorems in "General Algebra". Bull. Soc. Math. Grèce 28, 167–187 (1954). (Greek. English summary)

An algebra is a set Σ with ν μ-ary operations ϕ_λ. Basic concepts such as subalgebras, right and left unity elements, the commutative and associative laws, right and left unity elements, homomorphisms, and equivalence relations are defined for such systems. An equivalence relation R is called canonical with respect to a μ-ary operation ϕ if $\alpha_i=\beta_i$ (R), $i=1, 2, \cdots, \mu$, implies

$$\phi(\alpha_1, \alpha_2, \cdots, \alpha_\mu)=\phi(\beta_1, \beta_2, \cdots, \beta_\mu)\ (R).$$

Canonical equivalence relations are then used to prove the homomorphism theorem giving the correspondence between homomorphisms and equivalence relations. After discussing the special cases of groups and semi-groups, the concept of an algebra-ring is introduced. An algebra-ring is a set which is an abelian group with respect to each operation ϕ_i $(i=1, \cdots, \rho)$ and with a unity element common to all ϕ_i, a semi-group with respect to each f_j, $j=1, 2, \cdots, \sigma$, and each f_j is distributive with respect to each ϕ_i. The partitioning of algebra-rings by equivalence relations is studied and the homomorphism theorem for algebra-rings and the corresponding generalized ideals is proved. Boolean algebras are discussed as examples.

L. A. Kokoris.

Preston, G. B. Factorization of ideals in general algebras. J. London Math. Soc. 29, 363–368 (1954).

The author gives an extension of the Krull-Noether ideal theory in general algebras which he developed in a previous paper [same J. 29, 1–15 (1954); these Rev. 15, 390], to which the reader is referred for notation and basic definitions. Let $[A, \Omega]$ be a general algebra and $f_n\in\Omega$. If each a_i in $f_n(a_1, \cdots, a_n)$ is replaced by one of $s_1, \cdots, s_r$, where $r\leqq n$, the result is denoted by $f_n^r(s_1, \cdots, s_r)$. A set $\mathfrak{a}\subseteq A$ is prime$_r$ if $f_n^r(s_1, \cdots, s_r)\in\mathfrak{a}$ implies at least one $s_i\in\mathfrak{a}$, for all f_n^r and all $s_1, \cdots, s_r$ in A. A set $\mathfrak{a}$ is closed$_r$ with respect to f_n if $f_n^r(s_1, \cdots, s_r)\in\mathfrak{a}$ for all $s_1, \cdots, s_r\in\mathfrak{a}$. Hence a set $\mathfrak{a}$ is prime$_r$ if and only if $A-\mathfrak{a}$ is closed$_r$. All ideals considered are 1-ideals with respect to f_n. The radical$_r$ of an ideal, and a primary$_r$ ideal are suitably defined so that in either a ring algebra or an α-algebra, (1) the radical of a primary$_r$ ideal is prime$_r$; (2) the radical of an ideal $\mathfrak{a}$ is the intersection of the minimal prime$_r$ ideals containing $\mathfrak{a}$; (3) the Noether uniqueness theorems hold for ideals which can be represented as intersections of a finite number of primary$_r$ ideals. For $r=n$ the results reduce to those of the author's previous paper.

D. C. Murdoch (Vancouver, B. C.).

Verhoeff, J. Recent investigations about the radical of a ring. Math. Centrum Amsterdam. Rapport ZW 1953-007, 7 pp. (1953).

Expository paper.

Mařík, Jan. La réductibilité du déterminant ayant des indéterminées pour éléments, si l'on le considère comme un polynôme sur un anneau commutatif. Čehoslovack. Mat. Ž. 2(77), 279–293 (1952). (Russian. French summary)

Let O be a commutative ring with identities 0 and 1. Let R be the ring of polynomials in n^2 letters $x_{11}, x_{12}, \cdots, x_{nn}$ with coefficients in O. By D is meant the determinant of the $n\times n$ matrix whose ith row is $x_{i1}, \cdots, x_{in}$. Assuming that $D=F\cdot G$ with F and G in R, it is shown that there exist elements j_1, j_2 in O, and elements f_1, f_2, g_1, g_2 in R for which $f_1=f_1\cdot j_1$, $f_2=f_2\cdot j_2$, $g_1=g_1\cdot j_1$, $g_2=g_2\cdot j_2$, $f_1\cdot g_1=j_1$, $f_2\cdot g_2=j_2$, $j_1+j_2=1$, $j_1\cdot j_2=0$, $F=f_1+f_2D$, $G=g_2+g_1D$. It easily follows that $D=F\cdot G$ with F and G non-units in R if and only if O is a direct sum of two proper subrings.

F. J. Terpstra (Bandung).

Guérindon, Jean. Sur les idéaux minimaux dans les anneaux commutatifs. C. R. Acad. Sci. Paris **239**, 145–147 (1954).

Let A be a commutative ring with unity element and let L be the lattice of ideals of A. If, for $I \in L$, there exists $J \in L$ such that J covers I (i.e., $I<J$ but $I<K<J$ for no $K \in L$), then I is called quasi-uniform. Among the many elementary results on quasi-uniform ideals is the theorem that every nonzero ideal of A contains a minimal ideal if and only if $0:Au$ is quasi-uniform for every nonzero $u \in A$.

R. E. Johnson (Northampton, Mass.).

Zelinsky, Daniel. Raising idempotents. Duke Math. J. **21**, 315–322 (1954).

L'auteur étudie le problème suivant: étant donné un anneau R, un idéal bilatère N dans R, et une famille d'idempotents orthogonaux u_α dans R/N, est-il possible de "relever" ces idempotents en idempotents e_α de R, qui soient encore orthogonaux? Par un exemple ingénieux, il montre d'abord que la réponse peut être négative lorsque $N^2=0$, que R/N est somme directe de corps, mais que les U_α sont en infinité non dénombrable. Par contre, la réponse est positive si on fait intervenir des hypothèses supplémentaires de linéaire compacité: par exemple, si N est le radical de R, il suffit de supposer que R est une algèbre et que N est de dimension finie, ou que R est linéairement compact et que les adhérences des puissances de N ont une intersection réduite à 0. Ces résultats sont ensuite appliqués au problème de la généralisation du "théorème principal" de Wedderburn, affirmant l'existence d'une décomposition d'une algèbre (de dimension finie) séparable modulo le radical en somme directe d'une algèbre semi-simple et du radical. L'auteur montre ce théorème se généralise dans les cas suivants: (a) R est une algèbre linéairement compact, son radical N est tel que l'intersection des adhérences des N^k soit 0 et que R/N soit produit d'algèbres simples de dimension finie (cette hypothèse sur la dimension ne peut être omise); (b) R est compact et de caractéristique $p>0$ (p nombre premier). *J. Dieudonné* (Evanston, Ill.).

Behrens, Ernst-August. Nichtassoziative Ringe. Math. Ann. **127**, 441–452 (1954).

The author is concerned with the problem of defining a radical for a non-associative ring, with the usual requisite properties of a radical, and with the problem of then establishing a structure theory for the semi-simple case. If R is a (non-associative) ring, define a to be nilpotent if in some association a power of a is 0. An ideal is nil if all its elements are nilpotent. The author proves the existence of a maximal nil ideal and defines it to be his W-radical, $W(R)$. It is shown that $W(R/W(R))=(0)$. In case R possesses a unit element, then $W(R)$ is contained in the intersection of the maximal right ideals of R.

He then proceeds to define another radical $B(R)$, and the paper, in the main, is a study of this. If $a \in R$, let (a) be the smallest ideal of R containing a. Then $B(R)$ is defined as those elements a in R with the property that $b \in (b^2-b)$ for all $b \in (a)$. It follows that $B(R)$ has no idempotents and that $W(R) \subset B(R)$. Also $B(R/B(R))=(0)$, that is, is B-semi-simple. $B(R)$ is then characterized as $B(R)=\bigcap A_i$ over those ideals A_i of R such that in R/A_i the intersection of the non-zero ideals is generated by an idempotent e_i. Using this, the author demonstrates that the B-semi-simple rings are precisely those obtainable as subdirect sums of subdirectly irreducible rings whose minimal ideal is generated by an idempotent. The relation between $B(R)$ and a radical $S(R)$ introduced by Smiley [Amer. J. Math. **72**, 93–100 (1950); these Rev. **11**, 311] is considered.

I. N. Herstein (Philadelphia, Pa.).

Herstein, I. N. On the Lie ring of a simple ring. Proc. Nat. Acad. Sci. U. S. A. **40**, 305–306 (1954).

An announcement of results regarding the Lie ideals of any simple associative ring A with reference to the conjecture that if characteristic $A \neq 2$ the only proper Lie ideals of $[A, A]$ are those also in the center of A. It is stated that the conjecture is correct for division rings and in general if, in addition, characteristic $A \neq 3$ and A is not a nil ring. Additional information about the exceptional cases is given.

W. G. Lister (Providence, R. I.).

Koszul, J. L. Sur les modules de représentation des algèbres de Lie résolubles. Amer. J. Math. **76**, 535–554 (1954).

Let $H^n(g, M)$ denote the n-dimensional cohomology group for the Lie algebra g in the g-module M. Assume that g and M are of finite dimension over the base field k. A cohomology class $c \in H^n(g, M)$ is called effaceable if there exists a finite-dimensional g-module N containing M such that the canonical homomorphism $H^*(g, M) \to H^*(g, N)$ annihilates c. The main result of this paper is that if g is solvable and k of characteristic 0 then every element of $H^n(g, M)$ with $n>0$ is effaceable. The proof involves notions and auxiliary results which are of considerable interest by themselves.

Let $E(g)$ denote the universal enveloping algebra of g, and let $L(E(g), M)$ be the space of all k-linear maps f of $E(g)$ into M, made into a g-module by setting, for $x \in g$ and $e \in E(g)$, $(x \cdot f)(e)=f(ex)$. Let $L_f(E(g), M)$ be the submodule consisting of all those elements of $L(E(g), M)$ which generate finite-dimensional g-submodules. It is shown first that the effaceable cohomology classes are precisely those which are mapped into 0 by the canonical homomorphism of $H^*(g, M)$ into $H^*(g, L_f(E(g), M))$. Let $V(g)=L_f(E(g), k)$. Observing that $L_f(E(g), M)$ is g-isomorphic with the tensor product of $V(g)$ and M (in which M is treated as a trivial g-module), the effaceability problem is reduced to the study of $H^*(g, V(g))$. Concurrently with $V(g)$, the author considers also the submodule $W(g)$ consisting of those elements of $V(g)$ which generate g-submodules on which g is nilpotent. It is shown rather directly that, if g is abelian and k of characteristic 0, $H^n(g, V(g))=(0)=H^n(g, W(g))$ for all $n>0$. By using the method of spectral sequences, the following preliminary generalization of this result is obtained. Let k be of characteristic 0, and let b be an ideal of g such that g/b is abelian. Then there is a canonical isomorphism of $H^*(g, V(g/b))$ onto $H^*(b, k)$. Furthermore, if g is a nilpotent Lie algebra, there is a canonical isomorphism of $H^*(g, W(g/b))$ onto $H^*(b, k)$.

Using this result and suitable filtrations of $V(g)$, $W(g)$, etc. with respect to ideals of g, it is then shown inductively on the dimension of g that if g is nilpotent $H^n(g, W(g))=(0)$ for $n>0$. If g is solvable and b is the derived $[g, g]$ of g, it is shown by means of these filtrations that there is a canonical isomorphism of $H^*(g, V(g))$ onto $H^*(b, W(b))$, whence $H^n(g, V(g))=(0)$ for all $n>0$.

In the case where k is the real number field and g is the Lie algebra of a simply connected Lie group G, the author shows that the complex of cochains for g in $V(g)$ is canonically isomorphic with that of those differential forms on G whose translations make up finite-dimensional vector spaces. In this isomorphism, the cochains for g in $W(g)$

correspond to those differential forms whose translations make up spaces on which the infinitesimal transformations of the group of right translations are nilpotent.

G. Hochschild (Urbana, Ill.).

Hattori, Akira. Inner endomorphisms of an associative algebra. J. Math. Soc. Japan **6**, 40–44 (1954).

Let A be an algebra with identity and of dimension n over a field K, $K_0=K(X_1, \cdots, X_n)$, and

$$K_1=K(X_1, \cdots, X_n, Y_1, \cdots, Y_n),$$

where $X_1, \cdots, X_n, Y_1, \cdots, Y_n$ are independent variables over K. Extend A to the algebra A_1 over the field K_1. If (u_i) is a basis of A, then $S=\sum X_i u_i$ is a regular element of A_1. If $S(\sum Y_i u_i)S^{-1}=\sum Y_i' u_i$, then $Y_i'(X, Y)=\sum R_{ij}Y_j$, $R_{ij}(X) \in K_0$. The element $a=\sum \alpha_i u_i \in A$ is semiregular if each $R_{ij}(\alpha)$, obtained by replacing every X_k by α_k, is well-defined. The operation I_a: $\sum \beta_i u_i \to \sum Y_i'(\alpha, \beta)u_i$, a homomorphism of A into A, is called an inner endomorphism of A. Many elementary properties of inner endomorphisms are derived. *R. E. Johnson* (Northampton, Mass.).

Širšov, A. I. Subalgebras of free commutative and free anticommutative algebras. Mat. Sbornik N.S. **34(76)**, 81–88 (1954). (Russian)

This paper studies distributive but not associative algebras which satisfy either the commutative law $yx=xy$ or the anti-commutative law $yx=-xy$. Following the reviewer's paper [Proc. Amer. Math. Soc. **1**, 575–581 (1950); these Rev. **12**, 388], a basis is found for the free commutative and free anti-commutative algebras. Using a method similar to that of Kuroš [Mat. Sbornik N.S. **20(62)**, 239–262 (1947); these Rev. **9**, 5] and an earlier paper of the author [ibid. **33(75)**, 441–452 (1953); these Rev. **15**, 596] it is shown that the subalgebras of the free algebras are also free.

Marshall Hall, Jr. (Columbus, Ohio).

Rizza, Giovanni Battista. Sulla struttura delle algebre di Clifford. Rend. Sem. Mat. Univ. Padova **23**, 91–99 (1954).

On appele algèbre de Clifford C_n ($n=1, 2, \cdots$) l'algèbre réelle que l'on peut définir par une base

$$\{1, i_{\nu_1}, i_{\nu_1\nu_2}, \cdots, i_{\nu_1\nu_2\cdots\nu_n}\}, \quad \nu_1<\nu_2<\cdots<\nu_n=1, 2, \cdots, n,$$

avec les relations

$$i_{\nu_1}i_{\nu_2}\cdots i_{\nu_h}=i_{\nu_1\nu_2\cdots\nu_h}, \quad i_{\nu_h}^2=-1,$$
$$i_{\nu_h}i_{\nu_k}=-i_{\nu_k}i_{\nu_h} \quad (h<k=1, 2, \cdots, n),$$

où 1 désigne l'élément unité de C_n. L'auteur démontre les propositions suivantes. (a) L'algèbre C_n pour $n \geq 2$ est la somme directe de 2^{n-3} algèbres isomorphes à l'algèbre des quaternions. (b) Les diviseurs de zéro de C_n forment, dans l'espace R^{2^n} correspondant, 2^{n-2} sous-espaces linéaires de dimension 2^n-4. Ces résultats se relient aux recherches de l'auteur pour la généralisation de la théorie des fonctions analytiques ordinaires au cas des fonctions dans les systèmes hypercomplexes. *J. Sebastião e Silva* (Lisbonne).

Skornyakov, L. A. Concerning the note "On the theory of alternative fields." Uspehi Matem. Nauk (N.S.) **9**, no. 2(60), 185–188 (1954). (Russian)

It is shown that in an alternative ring of characteristic not three, if the associators of any three of the elements a, b, c, d are zero then also $(ab, c, d)=0$. This result is also true for a division ring of characteristic three, but a counter-example of characteristic three is given with 15 basis elements. Although reference is made to the basic paper of Bruck and Kleinfeld [Proc. Amer. Math. Soc. **2**, 878–890 (1951); these Rev. **13**, 526] the author does not realize that the direct result above appears in the appendix to this paper and also mention of a counterexample due to Smiley for the case of characteristic 3. *Marshall Hall, Jr.*

Northcott, D. G., and Rees, D. A note on reductions of ideals with an application to the generalized Hilbert function. Proc. Cambridge Philos. Soc. **50**, 353–359 (1954).

Les auteurs généralisent de la façon suivante la notion de réduction qu'ils avaient précédemment introduite [mêmes Proc. **50**, 145–158 (1954); ces Rev. **15**, 596]: soient $\mathfrak{o}$ un anneau local de dimension d, $\mathfrak{q}$ un idéal primaire ponr l'idéal maximal $\mathfrak{m}$ de $\mathfrak{o}$, $r(1), \cdots, r(d)$ des entiers ≥ 0; ou dit qu'un idéal $(v_1, \cdots, v_d)$ est une réduction de $\mathfrak{q}$ de type $(r(1), \cdots, r(d))$ si $v_i \in \mathfrak{q}^{r(i)}$ pour tout i, et s'il existe un entier $m \geq \max(r(1), \cdots, r(d))$ tel que $\mathfrak{q}^m=v_1\mathfrak{q}^{m-r(1)}+\cdots+v_d\mathfrak{q}^{m-r(d)}$. L'existence de réductions de type $(1, 1, \cdots, 1)$ de $\mathfrak{q}$ ne peut être assurée que si $\mathfrak{o}/\mathfrak{m}$ est infini [cf. loc. cit.]; on montre ici, par utilisation de la méthode des formes initiales, l'existence de réductions de type $(r, \cdots, r)$ de $\mathfrak{q}$, ou, ce qui revient au même, de type $(1, \cdots, 1)$ de $\mathfrak{q}^r$, pour certain entier r, et ceci sans hypothèse sur $\mathfrak{o}/\mathfrak{m}$. Par passage aux complétés on montre, dans le cas d'égales caractéristiques, que si $\mathfrak{v}=(v_1, \cdots, v_d)$ est une réduction de type $(r_1, \cdots, r_d)$ de $\mathfrak{q}$, les multiplicités de ces idéaux sont liées par $e(\mathfrak{v})=r_1\cdots r_d$; $e(\mathfrak{q})$. Enfin, et toujours dans le cas d'égales caractéristiques, on déduit du fait que $\mathfrak{q}^r$ admet des réductions de type $(1, \cdots, 1)$ que, si l'on note $\mathfrak{n}_i$ celles des composantes primaires de (0) qui sont de dimension d, on a

$$e(\mathfrak{q})=\sum_i e((\mathfrak{q}+\mathfrak{n}_i)/\mathfrak{n}_i);$$

en particulier la multiplicité d'un point P sur une variété algébrique V est égale à la somme des multiplicités de P sur les diverses nappes analytiques de V en P. *P. Samuel.*

Cohen, I. S. Lengths of prime ideal chains. Amer. J. Math. **76**, 654–668 (1954).

La longueur d'une chaine $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_k$ d'idéax premiers distincts d'un anneau R (commutatif avec élément unité) est l'entier k. On appelle hauteur (resp. profondeur) d'un idéal premier $\mathfrak{p}$ de R, et on note $h(\mathfrak{p})$ (resp. $d(\mathfrak{p})$), le maximum des longueurs des chaines d'idéaux premiers de R dont le dernier (resp. premier) terme est $\mathfrak{p}$. On s'occupe principalement ici des relations entre les hauteurs (resp. profondeurs) de deux idéaux premiers $\mathfrak{p}$, $\mathfrak{P}$ de deux anneaux d'intégrité R, S tels que $R \subset S$ et que $\mathfrak{p}=\mathfrak{P} \cap R$. On note $\mathrm{dt}(S:R)$ le degré de transcendance de S sur R, et $\dim(\mathfrak{P}/R)$ le degré $\mathrm{dt}(S/\mathfrak{P}:R/\mathfrak{p})$; le comportement de ces fonctions par passage aux anneaux quotients et aux anneaux de fractions est brièvement étudié; lorsque S est de type fini sur R et que $\mathfrak{p}=\mathfrak{P} \cap R=(0)$, alors $h(\mathfrak{P})+\dim(\mathfrak{P}/R)=\mathrm{dt}(S:R)$. L'auteur s'occupe alors de donner des conditions sous lesquelles l'inégalité

$$\mathrm{dt}(S:R)-\dim(\mathfrak{P}/R)+h(\mathfrak{p})-h(\mathfrak{P}) \geq 0 \tag{1}$$

est vraie. Son premier membre est additif, son comportement par passage aux anneaux quotients et aux anneaux de fractions est facile à étudier, et l'inégalité (1) résulte de théorèmes connus lorsque S est entier sur R. Lorsque R appartient à une classe d'anneaux d'intégrité, (a) stable par passage au quotient et adjonction d'indéterminées et, (b) telle que pour tout anneau A de cette classe et tout idéal premier minimal $\mathfrak{q}$ de A l'idéal $\mathfrak{q}A[X]$ soit un idéal premier minimal (par exemple la classe des anneaux d'intégrité

noethériens), alors (1) est vraie; la condition (b) peut être remplacée par la validité de (1) lorsque $\mathfrak{p}$ est un idéal premier minimal. Lorsque R est noethérien, que le corps des fractions de S est une extension de type fini de celui de R, et que (1) est une égalité, alors le corps des fractions de $S/\mathfrak{P}$ est une extension de type fini de celui de $R/\mathfrak{p}$. Les relations classiques entre le rang et la dimension des valuations d'un corps de fonctions algébriques sont des conséquences immédiates de ces résultats. Lorsque R appartient à une classe d'anneaux d'intégrité satisfaisant aux conditions (a) et (b) ci dessus, alors $d(\mathfrak{p})-d(\mathfrak{P})+\dim(\mathfrak{P}/R)\geqq 0$; de plus le premier membre est 0 ou 1 quand R est noethérien et S de type fini sur R; si de plus R est un anneau d'Hilbert (c.à.d. où tout idéal premier est intersection d'idéaux maximaux), ce premier membre est nul. L'amélioration des résultats donnés dans ce mémoire, ou bien se heurte à des contre exemples, ou bien dépend de "l'hypothèse des chaînes maximales"; celle ci, qui affirme que, pour tout idéal premier $\mathfrak{p}$ d'un anneau d'intégrité, toute chaine maximale d'idéaux premiers de dernier terme $\mathfrak{p}$ a pour longueur $h(\mathfrak{p})$, n'est pour l'instant vérifiée que pour des classes assez restreintes d'anneaux d'intégrité noethériens, comprenant cependant tous ceux que l'on rencontre naturellement en Géométrie Algébrique et en Arithmétique. *P. Samuel.*

*Ehrlich, Gertrude. **The structure of continuous rings.** Thesis, University of Tennessee, 1953. vii+81 pp. (mimeographed)

An abstract of this thesis appeared in Bull. Amer. Math. Soc. 60, 138–139 (1954). The author considers a ring R, assumed to be the von Neumann regular ring of a projective or continuous geometry, and the group G consisting of the elements of R which possess inverses in R. She shows that elements of class 2 (i.e. $s=1+n$ with $n^2=0$) can be characterized within G and hence deduces the theorem: G_1 is isomorphic to G_2 if and only if R_1 is either isomorphic or anti-isomorphic to R_2; every isomorphism of G_1 onto G_2 is the product of an isomorphism induced by a unique isomorphism or anti-isomorphism of R_1 onto R_2 followed by a singular automorphism of G_2.

This generalizes in part the same theorems proved by R. Baer for the case R=ring of linear mappings of a vector module (over a division ring) into itself. The present thesis follows Baer's methods [Linear algebra and projective geometry, Academic Press, New York, 1952, Chap. VI; these Rev. 14, 675] very closely but succeeds in putting the entire work in vector-free form, using theorems of von Neumann to supplement the machinery available from Baer's work. The author poses the problem of obtaining a treatment that applies both to Baer's case and to her case; actually, with minor modification, her treatment does include Baer's case *I. Halperin* (Utrecht).

Theory of Groups

Clifford, A. H. Naturally totally ordered commutative semigroups. Amer. J. Math. 76, 631–646 (1954).

An order-relation is defined in a commutative semigroup as follows: $a<b$ if there is a c for which $ac=b$, but no d such that $bd=a$. If this is a relation of total order, the semigroup is said to be "naturally totally ordered" (n.t.o.). Hölder showed [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Cl. 53, 1–64 (1901)] that a similar type of semigroup-with-cancellation was, if archimedean, embeddable in the semigroup-with-cancellation P of all positive real numbers. The present paper deals with the analogous result where the cancellation law is broken. The main result is that any archimedean n.t.o. semigroup S in which the cancellation law is broken can be extended uniquely to an archimedean n.t.o. semigroup-with-cancellation T, i.e., T has a quotient semigroup isomorphic with S. It follows that either S can be embedded in $P/[1]$ or S can be embedded in $P/(1)$, where (1) is the set of all real numbers greater than 1, and $[1]$ is the set of all real numbers not less than 1. An interesting preliminary result is that, if the ordinal sum of an ordered set of n.t.o. semigroups is defined in the natural way, then every n.t.o. semigroup is uniquely the ordinal sum of ordinally irreducible n.t.o. semigroups. *H. A. Thurston.*

Clifford, A. H. Bands of semigroups. Proc. Amer. Math. Soc. 5, 499–504 (1954).

A semigroup S is called a band if it is the class sum of disjoint semigroups the product of any two of which is contained in a third. If, for every pair S_α, S_β of the disjoint semigroups, $S_\alpha S_\beta$ and $S_\beta S_\alpha$ are contained in the same S_γ, the band is called a semilattice. If the set of suffixes is a cardinal product of two sets (i.e. if the semigroups can be furnished with double suffixes) and if $S_{i\kappa}S_{j\lambda}\subseteq S_{i\lambda}$ for all i, j, κ, λ; then the band is called a matrix. The main theorem is that a band of semigroups of a certain type τ is a semilattice of semigroups each of which is a matrix of semigroups of type τ. The proof is quite short, and depends on making the set of suffixes into a semigroup by putting $\alpha\beta=\gamma$, where γ is such that $S_\alpha S_\beta\subseteq S_\gamma$. The interest of this analysis lies in the possibility of displaying a given semigroup as a band of semigroups of a certain type. The author gives conditions for a given semigroup to be (i) a semilattice of completely simple semigroups, (ii) a semilattice of simple semigroups (without zeros), (iii) a band of groups, and (iv) a semilattice of groups. *H. A. Thurston* (Bristol).

Pierce, R. S. Homomorphisms of semi-groups. Ann. of Math. (2) 59, 287–291 (1954).

The author introduces and studies a new congruence relation for semi-groups (abbreviated s.g.). Notations: Let S denote a s.g. For a non-empty set $I\subset S$ (especially a non-empty (two-sided) ideal I of S) let $a\sim b$ (I) mean that $[(c,d)\,|\,cad\in I]=[(c,d)\,|\,cbd\in I]$. Here $(\sim)$ represents the mentioned congruence relation. S/I stands for the factoroid consisting of the congruence classes under $(\sim)$. An ideal I of S is called inclusive (in S) if whenever $cad\in I$ for all $c,d\in S$, then $a\in I$. A s.g. $\bar{S}$ is called disjunctive if it contains a zero element 0, and if, whenever $a\neq b$ in $\bar{S}$, $c,d\in\bar{S}$ exist so that either $cad=0$ and $cbd\neq 0$, or $cad\neq 0$ and $cbd=0$. Results: The main result consists of specializations of isomorphism theorems for groupoids [cf. Borůvka, Introduction to the theory of groups, 2d ed., Přírodovědecké Vydavatelství, Prague, 1952; these Rev. 15, 7] in the case of homomorphisms with relations to inclusive ideals and disjunctive semi-groups. For instance, the author's specialization of the first isomorphism theorem is the following one. If S is a s.g. and I an inclusive ideal of S, then S/I is a disjunctive semi-group. Conversely, if h is a homomorphism of S onto the disjunctive $\bar{S}$, then $I=h^{-1}(0)$ is an inclusive ideal of S and S/I is isomorphic to $\bar{S}$ by the mapping $[a]\to h(a)$. In the remainder of the paper the s.g. is specialized to a distributive lattice L and the congruence is used to study the homomorphisms of L. *O. Borůvka* (Brno).

Scott, W. R. On the order of the automorphism group of a finite group. Proc. Amer. Math. Soc. 5, 23–24 (1954).

The author conjectures that there is a function f such that if p is a prime number and $p^{f(n)}$ divides the order of a group, then p^n divides the order of the group of automorphisms of that group. He proves quite simply that $f(2)=3$. It is already known [I. N. Herstein and J. E. Adney, Amer. Math. Monthly 59, 309–310 (1952); these Rev. 13, 907] that $f(1)=2$. *H. A. Thurston* (Bristol).

Neumann, B. H. Groups covered by permutable subsets. J. London Math. Soc. 29, 236–248 (1954).

The main result of this paper characterizes the groups which can be expressed as the union of permutable boundedly finite subsets as precisely the groups which possess a subgroup of finite index having a finite derived group. The sufficiency is proved by a relatively simple direct argument. For the necessity the author first proves that the classes of conjugate elements of a group H are boundedly finite if and only if the derived group H' is finite. Let G be the union of permutable boundedly finite subsets. The author proves that the union H of the finite classes of conjugate elements is a subgroup of finite index whose classes of conjugate elements are boundedly finite so that H' is finite. An intermediate result states that if a group is the union of a finite number n of cosets $C_i g_i$ then at least one of the subgroups C_i has index $\leqq n$. A further application of this result is a purely group-theoretical proof of the following precise form of an unpublished theorem of F. I. Mautner: If K is a finite subgroup of a group G whose double cosets permute, then $G=HK$, where H is the union of the finite classes of conjugate elements of G. The author also gives some results about groups which are unions of permutable [not necessarily boundedly] finite subsets, e.g., every finitely generated group and every countable locally finite group, but no non-denumerable locally free group, has this property [see also the following review]. *D. G. Higman.*

Cohn, P. M. A countably generated group which cannot be covered by finite permutable subsets. J. London Math. Soc. 29, 248–249 (1954).

The author proves that the free group on a countably infinite set of generators cannot be expressed as a union of permutable finite subsets [see the preceding review]. *D. G. Higman* (Missoula, Mont.).

Haimo, Franklin. Automorphisms generated by a class of subnormal subgroups. Duke Math. J. 21, 349–353 (1954).

Let G be a group, and C a chain $G=G_0, G_1, \cdots, G_k, G_{k+1}=1$ of subgroups of G, each normal in the preceding. The author studies the group $A(C)$ of those automorphisms of G which are admitted by each group G_i and which induce the identical automorphism in each factor G_i/G_{i+1}. For instance, $A(C)$ is solvable with derived length at most k, and if G_k contains the centre of every group G_i, with the possible exception of that of G itself, then $A(C)$ is nilpotent of class at most k. *Graham Higman* (Manchester).

Fox, Ralph H. Free differential calculus. II. The isomorphism problem of groups. Ann. of Math. (2) 59, 196–210 (1954).

A presentation $(x; r)$ of a group G is a set of generators $(x_1, x_2, \cdots)$ together with a collection $r=(r_1, r_2, \cdots)$ of elements of X, the free group generated by $(x_1, x_2, \cdots)$. Here G is defined as X/R where R is the smallest normal subgroup R of X containing the r_i. To every presentation $(x; r)$ of G the author assigns a matrix $(\partial x/\partial r)$ over the integral group ring JG of G. This "Jacobian" of the presentation is the ϕ-image of the matrix with components $\partial x_i/\partial r_j$ [see R. H. Fox, Ann. of Math. (2) 57, 547–560 (1953); these Rev. 14, 843] in JX under the natural homomorphism ϕ: $JX \to JG$. The author defines five types of "elementary operations" on matrices over a ring R. They are, roughly, permutation of rows (or columns), adding of new rows (or columns) which are left linear combinations of the rows (or columns), the operation $A \to \begin{Vmatrix} A & 0 \\ 0 & 1 \end{Vmatrix}$.

Two matrices are equivalent if one can be transformed into the other one by a finite number of these elementary operations. The author shows that the Jacobians of two presentations of G are in the same equivalence class over JG. If A is a matrix over a commutative ring R the dth elementary ideal $E_d(A)$ is the ideal generated by the minor determinants of order $n-d$ (n=number of columns; $E_d(A)=(R)$ for $d \geqq n$). These ideals are shown to be invariant under equivalence as defined above.

Consider now the Abelian group-ring JH of $H=G/[G, G]$. Let χ: $JG \to JH$ be the natural homomorphism. χ maps the Jacobians of the presentation of G into equivalent matrices over JH and hence the elementary ideals $E_d(G)$ of the χ-image of any such presentation are actually invariants of G. An immediate corollary is the following necessary condition for the isomorphisms of two groups: $G^{(1)} \approx G^{(2)}$ only if $H^{(1)} \approx H^{(2)}$ and some isomorphism of $H^{(1)}$ upon $H^{(2)}$ transforms the elementary ideals of $G^{(1)}$ into those of $G^{(2)}$.

As examples of the theorems proved concerning the $E_d(G)$ we state the following:

$$E_d(G^{(1)} * G^{(2)}) = \sum_{d_1+d_2=d} E_{d_1}(G^{(1)}) \cdot E_{d_2}(G^{(2)})$$

($(G*H)$ means free product);

$$E_0(G) = \begin{cases} 0 & \text{if } H=G/[G, G] \text{ is infinite} \\ \sum_{h \in H} h & \text{if } H \text{ is finite.} \end{cases}$$

The author develops this whole theory for presentations of pairs (G, F), $F \subset G$. The final part of the paper describes how the cohomology groups of G can be computed from the Jacobian of a presentation $(x; r)$ of G. *R. Bott.*

Dieudonné, Jean. Les isomorphismes exceptionnels entre les groupes classiques finis. Canadian J. Math. 6, 305–315 (1954).

Among the classical groups of low dimensions there exist certain isomorphisms which depend only on the dimension and type of group considered but do not depend on the field of scalars of the vector space on which the groups operate. On the other hand, as has long been known, if the groups are finite there arise certain other isomorphisms not of the above type. The usual method of showing the existence of these exceptional isomorphisms is essentially nongeometric. In the present paper the author gives a more geometric treatment of the subject so that the exceptional isomorphisms appear in a somewhat more "natural" way than in the usual treatment. *C. E. Rickart.*

Frame, J. S., Robinson, G. de B., and Thrall, R. M. The hook graphs of the symmetric groups. Canadian J. Math. 6, 316–324 (1954).

A partition $(\lambda)=(\lambda_1, \lambda_2, \cdots, \lambda_m)$ of n, $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_m$, is associated with a graph of nodes containing λ_i nodes in the ith row, these nodes being in the first λ_i columns. The

node in the ith row, jth column together with all nodes in the same row and to the right of it, and all nodes in the same column and below it constitute a right hook. The number of nodes in the hook is denoted by h_{ij}. If each node in the diagram is replaced by the corresponding integer h_{ij}, then the hook-graph $H[\lambda]$ of the partition is obtained. This paper concerns the use of this hook graph in elucidating the properties of the irreducible representation $[\lambda]$ of the symmetric group on n symbols, particularly in connection with the star diagrams, p-residues and p-cores of the representation for various integers p.

Firstly the degree of the representation is $n!/H_\lambda$, where H_λ, the hook-product, is the product of all the integers h_{ij}. This result holds for the reducible representations corresponding to diagrams consisting of a number of disjoint right diagrams. In the hook graph of a partition (λ) delete all the numbers h_{ij} except those which are divisible by p. Divide each remaining h_{ij} by p. Thus is obtained the hook graph of the p-quotient (or star) diagram. If $[\lambda]_p$ denotes the p-quotient of $[\lambda]$, then $[[\lambda]_p]_q=[\lambda]_{pq}$. The p-exponent of the degree of the representation $[\lambda]$ is also discussed and a method given for determining the p-core.

D. E. Littlewood (Bangor).

*Raikow, D. A. **Die harmonische Analyse auf kommutativen Gruppen mit Haarschem Mass und die Theorie der Charaktere.** Sowjetische Arbeiten zur Funktionalanalysis, pp. 11–87. Verlag Kultur und Fortschritt, Berlin, 1954.

Translated from Trudy Mat. Inst. Steklov **14** (1945); these Rev. **8**, 133.

Schöneborn, Heinz. Über gewisse Topologien in Abelschen Gruppen. I. Math. Z. **59**, 455–473 (1954).

This paper and its successor (reviewed below) constitute a detailed study of topological abelian groups with subgroup neighborhoods of 0, from the point of view established by Prüfer [Math. Z. **22**, 222–249 (1925)] and several later authors (Pietrkowski, Krull, Vilenkin). Certain axioms are imposed concerning the behavior of cyclic subgroups and the existence of a suitable ring of operators. One of the consequences is that the theory may for the most part be reduced to the primary case. The primary groups in question have the following properties: the closure of any cyclic subgroups is complete, and modulo any open subgroup one gets an ordinary primary group. As a consequence they admit the p-adic integers as a ring of operators. Topics studied include the comparison of topologies on a group and the relations between a group and its subgroups.

I. Kaplansky (Chicago, Ill.).

Schöneborn, Heinz. Über gewisse Topologien in Abelschen Gruppen. II. Math. Z. **60**, 17–30 (1954).

The author takes up the question of duality for a primary group G of the type described in the preceding review. The character group G^* is defined to be the group of all continuous homomorphisms of G into the group $Z(p^\infty)$ of all p^nth roots of 1, the latter carrying the discrete topology; G^* is awarded the compact-open topology. If G is complete, or if it satisfies the weaker condition that any closed subgroup generated by a compact set is compact, then $G=G^{**}$ algebraically. It is conjectured that G is also homeomorphic to G^{**}, and this is proved under various weak additional assumptions. The special case of this duality where G is linearly compact was independently obtained by the reviewer [Proc. Amer. Math. Soc. **4**, 213–219 (1953); these Rev. **14**, 720].

I. Kaplansky (Chicago, Ill.).

Leptin, Horst. Über eine Klasse linear kompakter abelscher Gruppen. I. Abh. Math. Sem. Univ. Hamburg **19**, no. 1–2, 23–40 (1954).

This paper covers territory, essentially known, concerning linearly compact abelian groups, e.g. subgroups, products, decomposition into primary components, characterization of the compact case. The author reports that the papers of Schöneborn reviewed above came to his attention after the paper was complete. Also relevant are a paper by the reviewer [Proc. Amer. Math. Soc. **4**, 213–219 (1953); these Rev. **14**, 720] and a long series of papers by Vilenkin. The table of contents covers Part II as well, and it appears that the high spot will be a theory of duality. *I. Kaplansky.*

Kac, G. I. Isomorphic mapping of topological groups into a direct product of groups satisfying the first countability axiom. Uspehi Matem. Nauk (N.S.) **8**, no. 6(58), 107–113 (1953). (Russian)

Graev [Uspehi Matem. Nauk (N.S.) **5**, no. 2(36), 3–56 (1950); these Rev. **12**, 78] has shown that if a topological group has an invariant basis, it can be isomorphically imbedded in a direct product of first countable groups with the same property. Using similar arguments, the author proves that a topological group G can be isomorphically imbedded in a direct product of first countable groups if and only if it has a quasi-invariant basis, by which is meant that for every neighbourhood V of the identity in G, there can be found a countable system of neighbourhoods $V_i \subseteq V$ with the property that for every g there exists an i such that $g^{-1}V_i g \subseteq V$. This theorem is used to prove the following: every connected, locally compact group is isomorphic with a closed subgroup of a direct product of connected, locally compact, first countable groups. *J. L. Täts* (Brussels).

Vilenkin, N. Ya. On the classification of zero-dimensional locally compact abelian groups with an everywhere dense set of elements of finite order. Mat. Sbornik N.S. **34(76)**, 55–80 (1954). (Russian)

The purpose of this paper is to extend Ulm's theory for discrete, countable, primary abelian groups to a class of pairs $(G:H)$ where G is a locally compact 0-dimensional abelian group of type P (i.e. $p^n x \to 0$ as $n \to \infty$), and H is an open and compact subgroup of G. Similar theories have previously been obtained by the author [Mat. Sbornik N.S. **27(69)**, 85–102 (1950); **28(70)**, 503–536 (1951); **29(71)**, 13–30, 31–62 (1951); these Rev. **12**, 78; **13**, 110, 205, 206] for various classes of groups, or pairs, but none of them included Ulm's theory as a particular case.

The following notations are used:

$$p^n A=\{x: x=p^n y,\ y\in A\};\quad {}_n[A]=\{x: x\in A,\ p^n x=0\};$$
$${}_\infty[A]=\bigcup_n {}_n[A];$$

G_α and G_α^* are defined for every ordinal α by the relations $G_0=G_0^*=G$ and $G_\alpha=\bigcap_n \overline{p^n G_{\alpha-1}}$, $G_\alpha^*=\bigcap_n p^n G^*_{\alpha-1}$ or $G_\alpha=\bigcap_{\beta<\alpha} G_\beta$, $G_\alpha^*=\bigcap_{\beta<\alpha} G_\beta$, according to whether α does or does not have a predecessor; τ is the smallest α for which $G_\alpha=G_{\alpha+1}$; $\hat{G}_\alpha=G_\alpha/G_{\alpha+1}$; $\hat{H}_\alpha$ is the image of $H_\alpha=H\cap G_\alpha$ under the natural homomorphism of G_α onto $\hat{G}_\alpha$. The pairs $(G:H)$ studied in this paper are those for which the following relations hold identically in m, n, α:

$$p^n G\cap \overline{p^{n+m}G_\alpha}=p^n\overline{[p^m G_\alpha]},\quad {}_n\overline{[p^m G_\alpha]}={}_n\overline{[p^m G_\alpha^*]},$$
$$p^n G\cap G_\alpha=p^n G_\alpha+G_\alpha^*,\quad G_\alpha={}_\infty[\hat{G}_\alpha],$$
$$p^n H\cap p^{n+m}G_\alpha^*=p^n(H\cap p^m G_\alpha^*),\quad p^n H\cap G_\alpha^*=p^n(H\cap G_\alpha^*).$$

The following theorems are proved for pairs of that type: Th. 1: For every $\alpha<\tau$, $(\hat{G}_\alpha:\hat{H}_\alpha)$ is a direct sum [Vilenkin,

ibid. 19(61), 85–154 (1946); these Rev. 8, 132] of cyclic groups and $(G_r : H_r)$ is a direct sum of groups of type p^∞. Th. 2: A countable direct sum of pairs of the considered type is also of that type. Th. 3 states necessary and sufficient conditions for a priori given pairs $(\hat{G}_\alpha : \hat{H}_\alpha)$ to be the "Ulm factors" of a pair $(G : H)$ (rather obvious necessary conditions appear to be also sufficient). Th. 4: If $G_r = 0$, $(G : H)$ is completely determined by the pairs $(\hat{G}_\alpha : \hat{H}_\alpha)$. Th. 5: $(G_r : H_r)$ is a direct summand of $(G : H)$. *J. L. Tits.*

Iséki, Kiyoshi. On compact abelian semi-groups. Michigan Math. J. 2, 59–60 (1954).

The author extends an algebraic result of Thierrin [C. R. Acad. Sci. Paris 236, 33–35 (1953); these Rev. 14, 616] by showing that the kernel (=minimal ideal) of a compact abelian topological semigroup is a group. This result is also contained in more general form in an abstract of the reviewer [Bull. Amer. Math. Soc. 60, 56–57 (1954)].

R. J. Koch (Baton Rouge, La.).

Tamura, Takayuki. On compact one-idempotent semigroups. Kōdai Math. Sem. Rep. 1954, 17–21 (1954).

The words "sequentially compact" and "compact" will be used for the author's "compact" and "bicompact". The preliminary results of the note are supposed to prove for sequentially compact Hausdorff semigroups some results that are known in the compact case. The author should have included references to A. H. Clifford and D. D. Miller [Amer. J. Math. 70, 117–125 (1948); these Rev. 9, 330] and J. E. L. Peck [Proc. Amer. Math. Soc. 2, 414–421 (1951); these Rev. 12, 835]. The latter part of the paper is devoted to a proof of the fact that, in a sequentially compact Hausdorff semigroup S with only one idempotent, the intersection of the sets $\{S^n \mid n = 1, 2, \cdots\}$ is the minimal ideal of S and hence if $S^2 = S$ then S is a topological group. For the compact case this result was proved by Koch and Wallace in a paper to be published later. The reviewer was not always able to recover the arguments which were omitted or follow those that were given and presumes that the author was tacitly assuming a countability axiom. The author promises a paper on sequentially compact monothetic semigroups, a subject treated by R. J. Koch in the compact case in his dissertation [Tulane, 1953]. *A. D. Wallace.*

Wallace, A. D. Indecomposable semigroups. Math. J. Okayama Univ. 3, 1–3 (1953).

A mob is a topological (Hausdorff) semigroup. A clan is a compact, connected mob with unit. This paper continues earlier work of the author [Anais Acad. Brasil. Ci. 24, 329–334 (1952); these Rev. 14, 724] and Numakura [Math. J. Okayama Univ. 1, 99–108 (1952); these Rev. 14, 18] on the structure of compact mobs. The main contributions of this paper are two sufficient conditions for a clan to be a topological group; namely that the clan be either (i) a metric indecomposable continuum [Moore, Foundations of point set theory, Amer. Math. Soc. Colloq. Publ., v. 13, New York, 1932] or (ii) a (classical) manifold. [The proof of (ii) is not in this paper, see Wallace, Summa Brasil. Math. 3, 43–55 (1953); these Rev. 15, 336.] Examples are given and some interesting problems are presented.

M. Henriksen (Lafayette, Ind.).

Wallace, A. D. Inverses in Euclidean mobs. Math. J. Okayama Univ. 3, 23–28 (1953).

The main contribution of this paper is a proof that if a clan S (for definitions and background material, see the preceding review) is contained in n-dimensional Euclidean space, $n \geqq 2$, then all elements with inverse lie on the boundary of S. Some of the intermediate results are of independent interest, but are too complex to give in a short review. The proofs rely on the material cited in the above review, some (unpublished) results of Koch, and a paper of Spanier on cohomology theory [Ann. of Math. (2) 49, 407–427 (1948); these Rev. 9, 523]. *M. Henriksen.*

NUMBER THEORY

***Vinogradov, I. M. Elements of number theory.** Translated by S. Kravetz. Dover Publications, Inc., New York, 1954. viii+227 pp. Cloth $3.00; paper $1.75.

Translation of the 5th edition of the author's Osnovy teorii čisel [Gostehizdat, Moscow, 1949; these Rev. 12, 10].

Gabard, E. Sur deux factorisations. Mathesis 63, 117–119 (1954).

Cavallaro, Vincenzo G. Equazioni diofantee brocardiane. Giorn. Mat. Battaglini (5) 2(82), 301–308 (1954).

Natucci, Alpinolo. Ricerche sistematiche intorno al "teorema di Catalan". Giorn. Mat. Battaglini (5) 2(82), 297–300 (1954).

Catalan's theorem states that 8 and 9 are the only powers of positive integers which differ by unity. No proof or disproof is known. The author describes and illustrates a somewhat pedestrian method of systematic search for two such powers. He concludes that the probability of the truth of Catalan's assertion is very great. *D. H. Lehmer.*

Duparc, H. J. A. On Mersenne numbers and Poulet numbers. Math. Centrum Amsterdam. Rapport ZW 1953-001, 2 pp. (1953).

A Poulet number (or pseudo prime) is defined as a composite number m which divides $2^m - 2$. In case every divisor d of a Poulet number N divides $2^d - 2$ then N is called a super Poulet number. Every composite Mersenne number is a super Poulet number. The sequence m_n defined recursively by $m_{n+1} = 2^{m_n} - 1$ $(n = 0, 1, \cdots)$, in which m_0 is a prime, either consists wholly of primes or else, for some k, m_n is a prime or a Poulet number according as $n \leqq k$ or $n > k$.

D. H. Lehmer (Berkeley, Calif.).

Duparc, H. J. A. On Carmichael numbers, Poulet numbers, Mersenne numbers and Fermat numbers. Math. Centrum Amsterdam. Rapport ZW 1953-004, 7 pp. (1953).

Expository paper.

***Worobjow, N. N. Die Fibonaccischen Zahlen.** Deutscher Verlag der Wissenschaften, Berlin, 1954. 47 pp.

Translation of the author's popular booklet, Čisla fibonačči [Gostehizdat, Moscow, 1951; these Rev. 14, 1062].

Reichman, Raphael I. A summation formula involving Fibonacci numbers. Scripta Math. 20, 111–112 (1954).

Let $s_k(n) = \sum_{i=1}^{n} i^k a_i$ where k is a positive integer and a_i is the ith term of the Fibonacci series

$$(a_1 = a_2 = 1,\ a_{i+1} = a_{i+1} + a_i).$$

It is stated that

$$s_k(n)=\sum_{i=0}^{k}(-1)^i a_{n+i+2}\Delta^i n^k+c_k,$$

where the constant c_k is to be determined from the fact that $s(0)=0$. This result is proved for $k=1, 2, 3$; the general proof is not given although this is easily accomplished by induction. *H. W. Brinkmann* (Swarthmore, Pa.).

Ankeny, N. C., and Erdös, P. The insolubility of classes of diophantine equations. Amer. J. Math. **76**, 488–496 (1954).

Proofs of the following two theorems. Theorem I. Let $a_1, \cdots, a_n$ be rational integers and suppose that

$$e_1a_1+\cdots+e_na_n=0,$$

with $e_j=0$ or ± 1, implies $e_1=e_2=\cdots=e_n=0$. Then the Diophantine equation

$$a_1X_1^n+\cdots+a_nX_n^n=0$$

has a non-trivial rational integer solution for almost no integers $m>0$ (in the sense that the number of $m\leqq M$ for which a solution exists is $o(M)$ as $M\to\infty$). Theorem II. The equation $X_1^m+X_2^m+X_3^m$ has a rational integer solution with g.c.d. $(X_1X_2X_3, m)=1$ for almost no integers $m>0$. The argument extends to general algebraic number fields, and is stated to be capable of replacing $o(M)$ by an explicit estimate $O(M(\log M)^{-c})$ for some $c>0$. The reviewer remarks that the methods appear to say little about the relative density of the primes $p=m$, for which the equations are soluble, in the set of all primes. *J. W. S. Cassels.*

Carlitz, L., and Olson, F. R. Some theorems on Bernoulli and Euler numbers of higher order. Duke Math. J. **21**, 405–421 (1954).

Nörlund [Vorlesungen über Differenzenrechnung, Springer, Berlin, 1924] has defined generalized Bernoulli numbers $B_m^{(k)}[\omega_1, \cdots, \omega_k]$ by means of

$$\prod_{s=1}^{k}\left(\frac{\omega_s x}{e^{\omega_s x}-1}\right)^{\epsilon}=\sum_{m=0}^{\infty}B_m^{(k)}[\omega_1, \cdots, \omega_k]\frac{x^m}{m!}$$

in which $\epsilon=1$ for the numbers of positive order and $\epsilon=-1$ for those of negative order. Let $\omega_1, \cdots, \omega_k$ be rational numbers that are integral (mod p), where p is a fixed prime >3. The main object of the present paper is to extend well-known congruence formulas for Bernoulli numbers to the generalized numbers. For example, the authors show that if $\sigma_s=\omega_1^s+\cdots+\omega_k^s\equiv 0 \pmod p$ $(1\leqq s<p-1)$ for some fixed k, then

$$B_{2r}^{(k)}[\omega_1, \cdots, \omega_k]\equiv -\frac{1}{2r}\sigma_{2r}B_{2r} \pmod{p^2}$$

for $1\leqq r\leqq\frac{1}{2}(p-3)$. Similar congruences are obtained for generalized Euler numbers and other analogous numbers. These results are derived as special cases of like formulas for certain numbers of "mixed order" defined in terms of two sets of parameters $\omega_1, \cdots, \omega_k$, $\omega_1^*, \cdots, \omega_k^*$. In addition, reciprocity formulas are obtained. Finally, the authors consider the case of more general sequences, thus extending most of their previous results. *A. L. Whiteman.*

Carlitz, L. Congruences for the number of n-gons formed by n lines. Amer. Math. Monthly **61**, 407–411 (1954).

Let g_n denote the number of polygons of n sides (including degenerate cases) formed by a network of n lines. If $m>1$ and $m_0=m$ for m odd and $m_0=m/2$ for m even, the author shows: $g_{n+m}\equiv g_ng_m \pmod{m_0}$. If

$$\Delta^r g_n=\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}g_{n+s}g_{(r-s)n}$$

then $\Delta^r g_n\equiv 0 \pmod{m_0^{[r+1]/2}}$. Furthermore if $m=p$ is an odd prime then $g_p\equiv\frac{1}{2}(p-1) \pmod p$. *H. B. Mann.*

Sierpiński, W. Remarques sur les racines d'une congruence. Ann. Polon. Math. **1**, 89–90 (1954).

Chojnacka has proved (in a paper that has not yet been published) that if m is composite $\neq 4$, there exist three numbers a, b, c, no two of which are congruent (mod m), such that if $f(x)$ is a polynomial with integral coefficients and $f(x)\equiv f(b)\equiv 0 \pmod m$ then $f(c)\equiv 0 \pmod m$. The present paper contains a proof of the following stronger result. If m is composite $\neq 4$, there exist two integers a, b, $a\not\equiv 0$, $b\not\equiv 0 \pmod m$, such that if $f(x)$ is a polynomial with integral coefficients and $f(a)\equiv f(b)\equiv 0 \pmod m$ then $f(0)\equiv 0 \pmod m$. As a corollary, there exists a quadratic polynomial $f(x)=x^2+bx+c$ such that $f(x)\equiv 0 \pmod m$ has more than two roots. *L. Carlitz* (Durham, N. C.).

Salié, Hans. Zur Verteilung natürlicher Zahlen auf elementfremde Klassen. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Nat. Kl. **101**, no. 4, 26 pp. (1954).

Let

$$\sum_{v=1}^{n}a_{u,v}x_v, \quad u=1, 2, \cdots, m, \quad n>m \tag{1}$$

be a system of homogeneous equations with rational coefficients. Rado calls (1) regular, if for any division of the integers into k classes (k arbitrary) (1) is solvable in at least one of the classes. He calls (1) precisely l-regular if this holds for any $k\leqq l$, but fails for $k=l+1$, i.e., if there exists a division of the integers into $l+1$ classes so that in none of the classes is (1) solvable.

Rado conjectured that there exists an l_n so that if (1) is l_n-regular, then it is regular. He proved that it will suffice to prove this conjecture for $m=1$. The author proves this conjecture in several special cases; he also conjectures that $l_n<cn\log n$. If (1) is l-regular, there exists an s_l such that, if one splits the integers $\leqq s_l$ into l classes, (1) is solvable in at least one of the classes. The author determines s_l in some special cases, e.g., he shows that, for $2(x+y)=3z$, $s_3=54$. [Reviewer's remark: This later result was also proved by E. Straus in an unpublished letter to the reviewer.] *P. Erdös.*

van der Corput, J. G. On sums of systems. Math. Centrum Amsterdam. Scriptum no. **7**, 31 pp. (1954).

The author's main result is the following theorem: If $n\geqq 2$ and each of the systems $A_1, \cdots, A_n$ contains the number zero and A_n contains at least one positive number, then it is possible to construct $n-1$ systems $B_1, \cdots, B_{n-1}$ with the following properties: A_j $(1\leqq j\leqq n-1)$ is a subset of B_j, but $B_1+\cdots+B_{n-1}$ is a subset of $A_1+\cdots+A_n$. Each symmetric function $f(m; A_1, \cdots, A_n)$ satisfies for each positive m the inequality $f(m; B_1, \cdots, B_{n-1}, 0)\leqq f(m; A_1, \cdots, A_n)$. If k is positive and if a slowly changing function $\varphi(m)$ satisfies for each positive $m\leqq k$ the inequality

$$A_1(m)+\cdots+A_n(m)\geqq\varphi(m)+1-n,$$

then the inequality

$$B_1(m)+\cdots+B_{n-1}(m)\geqq\varphi(m)+1-n$$

holds for each positive $m \leqq k$. Here a system is a non-empty set of non-negative numbers, $A(m)$ denotes the number of positive integers in the system A which are smaller than m. A slowly changing function $\varphi(m)$ satisfies the inequality

$$\varphi(m+m') \leqq \varphi(m)+\varphi(m'),$$

an elementary symmetric function is a sum of terms of the form $(W+A_{i_1}+\cdots+A_{i_\nu})(m)$ symmetric in the indices, where W is any system and a symmetric function is a linear function with non-negative coefficients of elementary symmetric functions.

The reviewer's theorem [Ann. of Math. (2) **43**, 523–527 (1942); these Rev. **4**, 35] and a result of Dyson [J. London Math. Soc. **20**, 8–14 (1945); these Rev. **7**, 365] can be obtained from the author's result as special cases. The author's methods are very similar to those of Dyson. Several other theorems too complicated to be quoted here are also proved. The author's terminology is somewhat misleading. The $\alpha+\beta$ theorem is called Khintchine's theorem although Khintchine never proved it and the theorem of Dyson is not quoted in its full strength. *H. B. Mann* (Columbus, Ohio).

van der Pol, Balth. The representation of numbers as sums of eight, sixteen and twenty-four squares. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 349–361 (1954).

Ramanujan's formulas for $r_8(n)$, $r_{16}(n)$, $r_{24}(n)$ are derived in a relatively simple way. The author [same Proc. **54** = Indagationes Math. **13**, 261–271, 272–284 (1951); these Rev. **13**, 135] has already obtained the principal relations and the derivations depend on the skillful choice of certain modular transforms. The $r_{8k}(n)$ are the coefficients in the power series development of θ_3^{8k}. Some simplicity and symmetry is gained by considering θ_2^{8k}, θ_0^{8k}, θ_3^{8k} together. Formulas for the coefficients of θ_2^{8k} and θ_0^{8k} are also obtained for $k=1, 2, 3$. *H. S. Zuckerman* (Seattle, Wash.).

Prachar, K. On integers having many representations as a sum of two primes. J. London Math. Soc. **29**, 347–350 (1954).

Let $f(n)$ denote the number of representations of an even integer n as the sum of two primes. The author proves that, for sufficiently large x, there are more than $\exp(c_1\sqrt{(\log x)})$ even integers $n \leqq x$ for which $f(n) > (c_2 x \log\log x)/(\log x)^2$, where c_1 and c_2 are positive constants. The proof itself is elementary, but uses a formula due to Titchmarsh [Rend. Circ. Mat. Palermo **54**, 414–429 (1930); **57**, 478–479 (1933)] for the number of primes in an arithmetic progression.

R. D. James (East Lansing, Mich.).

Chevallier, J.-M. Essai de systématisation des méthodes concernant les nombres premiers. Bull. Soc. Roy. Sci. Liège **23**, 125–140 (1954).

This paper amplifies some of the results previously published by the same author [Rev. Gén. Sci. Pures Appl. (N.S.) **57**, 102–106 (1950); these Rev. **12**, 805]. Among the author's principal conclusions the following may be quoted: "La fonction arithmétique classique $\mu(n)$ et les sommes qui la contiennent comme $M(x)=\sum_{n \leqq x}\mu(n)$, se rattachent aisément et explicitement à l'ensemble des entiers lui-même, sans intervention des nombres premiers".

A. L. Whiteman (Los Angeles, Calif.).

Cugiani, Marco. Sulle "catene" di numeri primi consecutivi a differenze limitata. Ann. Mat. Pura Appl. (4) **36**, 121–132 (1954).

The author proves among others the following theorem: Let $n \to \infty$ and $c_1 < f(n) < c_1^{-1}\log n$, where c_1 is a suitable constant, and let $\eta > 0$ be an arbitrary positive constant. Then there exists a constant c_2 so that there are $c_2 f(n)$ consecutive primes $p_k, p_{k+1}, \cdots$ in the interval $((1-\eta)n, n)$ for which $p_{k+r+1}-p_{k+r} > \log n/f(n)$. *P. Erdös.*

Rodosskiĭ, K. A. On the least prime number in an arithmetic progression. Mat. Sbornik N.S. **34**(76), 331–356 (1954). (Russian)

This is a more detailed account of the author's previous paper [Doklady Akad. Nauk SSSR (N.S.) **88**, 753–756 (1953); see also Ukrain. Mat. Žurnal **3**, 399–403 (1951); these Rev. **15**, 202]. The following are the two main results. (1) Let D be an integer $\geqq 3$, ψ a real number between 2 and 0.1 ln D and $Q(D, \psi)$ the number of L-functions (formed with characters for modulus D) having at least one zero in the rectangle $1-\psi \ln^{-1} D \leqq \sigma \leqq 1$, $|t| \leqq e^{\psi} \ln^{-1} D$. Then $Q(D,\psi) < e^{A_1\psi}$, A_1 being a positive absolute constant. (2) Linnik's theorem: Let $P_{\min}(D, l)$ denote the least prime number in the progression $nD+l$, $(D, l)=1$. Then $P_{\min}(D, l) < D^{A_2}$, A_2 being a positive absolute constant. *W. H. Simons.*

Lekkerkerker, C. G. Prime factors of the elements of certain sequences of integers. Math. Centrum Amsterdam. Rapport **ZW 1953-003**, 17 pp. (1953).

Let a and b be non-zero rational integers with $a^2+4b>0$, and let ω and $\bar{\omega}$ be the roots of the equation $x^2-ax-b=0$. It is shown that the sequence of rational integers

$$u_n = (\omega^n-\bar{\omega}^n)/(\omega-\bar{\omega})$$

has the property that for each positive integer n, with a finite number of exceptions, there is a prime q such that $q \mid u_n$ but $q \nmid u_1 \cdots u_{n-1}$. This extends a recent theorem of T. Bang (apparently not yet published), where ω and $\bar{\omega}$ were required to be rational. It is shown in particular that each element of the Fibonacci sequence, defined by the equations $u_0=0$, $u_1=1$, $u_{n+2}=u_{n+1}+u_n$, contains a new prime divisor, except when $n=1, 2, 6$, or 12.

W. J. LeVeque (Ann Arbor, Mich.).

Val'fiš, A. Z. On the theory of prime numbers. Soobščeniya Akad. Nauk Gruzin. SSR **14**, 77–83 (1953). (Russian)

Under the assumption of the Extended Riemann Hypothesis (that is, the conjecture that the real parts of the zeros of the Dirichlet L-functions do not exceed $\frac{1}{2}$) the author proves that

$$\sum_{1<p<N} d(N-p) = \frac{315\zeta(3)}{2\pi^4} N \prod_{p \mid N} \frac{(p-1)^2}{p^3-p+1} + O\left(\frac{N(\log\log N)^2}{\log N}\right),$$

where N is a positive integer >2, p runs over the prime numbers, and $d(n)$ is the number of divisors of the positive integer n. The proof, which is straightforward and uses only well-known results, is similar to the proof given by Titchmarsh (also under the assumption of the E. R. H.) for a somewhat analogous asymptotic formula for $\sum_{1<p \leqq N} d(p-l)$, where l is a fixed non-zero integer [Rend. Circ. Mat. Palermo **54**, 414–429 (1930); **57**, 478–479 (1933)]. Remark: The reviewer does not know how to justify the assertion on the next to the last line of p. 78, namely, that if N is any positive integer, $M=N^{\frac{1}{2}}$, and n is any positive integer $\leqq M$, then $|\pi(N; n, N)-\pi(N-nM; n, N)-\pi(nM; n, N)|$ is less

The results of this paper are not new; they were obtained by R. D. Carmichael [Ann. of Math. (2) **15** (1913), ~~correct~~ ~~by R. D.~~ 30–70]. The paper by Bang mentioned in the ~~re~~review appeared in Tidsskr. f. Mat. (5) **4** (1886), 130–137.

than an absolute constant, $\pi(y; k, l)$ being the number of prime numbers not exceeding y and congruent to l modulo k. However, the proof can easily be modified so that the statement in question is not used. One need only replace $\pi(nM; n, N)$ in the statement of Lemma 2 and in its three occurrences on the third from the last line of p. 82 by $\pi(N; n, N)-\pi(N-nM; n, N)$ and also replace $\int_2^{nM} dz/\log z$ on the next to the last line of p. 82 by $\int_{N-nM}^{N} dz/\log z$.

P. T. Bateman (Urbana, Ill.).

Val'fiš, A. Z. On the representation of numbers by sums of squares. Asymptotic formulas. Uspehi Matem. Nauk (N.S.) 7, no. 6(52), 97–178 (1952). (Russian)

This paper is a very readable survey of the field in question. The author has taken great pains to give all the proofs in full detail and to make his treatment as self-contained as possible, with the result that it can be read by anyone who knows elementary number theory and the rudiments of analysis and algebra. The paper is divided into two main parts. The first is devoted to a very thorough treatment of the asymptotic formula for the number $r_k(n)$ of representations of a positive integer n as a sum of k squares ($k \geqq 5$), namely

$$r_k(n)=\frac{\pi^{k/2}}{\Gamma(k/2)}n^{k/2-1}\mathfrak{S}_k(n)+O(n^{k/4}),$$

where $\mathfrak{S}_k(n)$, the so-called singular series, is a bounded function of n given by a certain infinite series involving Gaussian sums. This result is the simplest application of the circle method of Hardy, Littlewood, and Ramanujan and as such has much didactic value in addition to its intrinsic interest. It was stated without proof by Hardy and by Ramanujan, but was first proved explicitly (in more general form) by Walfisz himself [Math. Z. 19, 300–307 (1924)]. The second and larger part of the paper is devoted to a similarly thorough treatment of the analogous asymptotic formula for the number $r_k(m, n)$ of solutions in integers of the pair of simultaneous equations

$$x_1+\cdots+x_k=m, \quad x_1^2+\cdots+x_k^2=n,$$

where k is a given integer greater than 7 and m and n are integers such that $m \equiv n \pmod 2$ and $kn-m^2>0$. The formula in this case is

$$r_k(m, n)=\frac{\pi^{(k-1)/2}}{k^{\frac{1}{2}k-1}\Gamma(\frac{1}{2}(k-1))}(kn-m^2)^{(k-4)/2}\mathfrak{S}_k(m, n) +O(n^{\frac{1}{2}k-2}\log n),$$

where $\mathfrak{S}_k(m, n)$ is somewhat similar to $\mathfrak{S}_k(n)$ above. The material discussed in this second part of the paper stems mostly from a paper of Kloosterman [Math. Ann. 118, 319–364 (1942); these Rev. 5, 33; 9, 735], a paper of van der Blij [Nederl. Akad. Wetensch., Proc. 50, 31–40, 41–48 = Indagationes Math. 9, 16–25, 26–33 (1947); these Rev. 8, 502], and a paper by the author [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 19, 33–59 (1953)] and one by G. A. Lomadze [ibid. 19, 61–77 (1953)]. In both parts of the paper the singular series is discussed in great detail, but no exact results are proved for either $r_k(n)$ or $r_k(m, n)$.

P. T. Bateman (Urbana, Ill.).

Malyšev, A. V. An asymptotic law for the representation of numbers by some positive ternary quadratic forms. Doklady Akad. Nauk SSSR (N.S.) 93, 771–774 (1953); erratum, 95, 700 (1954). (Russian)

Let f be a positive, integral, properly primitive, ternary quadratic form of invariants $[r, 1]$, r odd, with characters $(b/p)=(-1)^{p-1}$ for all primes $p \mid r$. For any integer m prime to r, denote by $N(f, m)$ the number of primitive representations of m by f. Asymptotic formulas for $N(f, m)$ as $m \to \infty$ are obtained in terms of the prime divisors of r and the number of classes of positive properly primitive binary quadratic forms of determinant m. The proof, employing the arithmetic of quaternions, is related to work of Linnik [Izvestiya Akad. Nauk SSSR. Ser. Mat. 4, 363–402 (1940); these Rev. 2, 348].

I. Niven (Eugene, Ore.).

Jones, Burton W., and Marsh, Donald. Automorphs of quadratic forms. Duke Math. J. 21, 179–193 (1954).

A étant la matrice d'une forme quadratique non-dégénérée (c'est donc une matrice symétrique) à coefficients dans un corps F de caractéristique $\neq 2$, un procédé explicite de construction de tous les automorphes T de A (autrement dit, de toutes les matrices telles que $T'AT=A$, où T' est la transposée de T), exprimables comme polynômes à coefficients dans F en $L=T-T^{-1}$, est donné. La construction est définie quand on se donne une matrice antisymétrique R de même ordre que A et deux polynômes $h_1(x)$, $h_2(x) \in F[x]$ satisfaisant à certaines conditions explicites, qui ne font intervenir que la structure matricielle de A. Mais un même automorphe de A peut s'obtenir, par cette construction, à partir des R, h_1 et h_2 différents. En vue d'obtenir chaque automorphe T une seule fois, on normalise cette construction en imposant la condition $L=T-T^{-1}=RA/d$, où d est le déterminant de A, mais cette normalisation s'exprime d'une manière peu explicite en termes des R, h_1, h_2.

La forme des automorphes T tels que $L=0$ (autrement dit, tels que $T^2=I$) est également déterminée, ainsi que les conditions, auxquelles A doit satisfaire pour posséder de tels automorphes; et, dans le cas, où F est le corps rationnel, les conditions nécessaires et suffisantes sont données pour que A possède un tel automorphe à éléments entiers. Comme application, tous les automorphes à éléments entiers rationnels d'une forme quadratique ternaire à coefficients entiers sont determinés.

M. Krasner (Paris).

Oppenheim, A. On indefinite binary quadratic forms. Acta Math. 91, 43–50 (1954).

Let $Q(x, y)$ be an indefinite quadratic form of discriminant D^2. A theorem of Barnes states that there are integers u, v, x, y such that

$$(*) \quad -D^2(2\sqrt{5}+2)^{-1} \leqq Q(x, y)Q(u, v)<0, \quad xv-yu=\pm 1$$

except when Q is equivalent to one of a denumerable set of forms; but this is no longer true if $2\sqrt{5}+2$ is replaced by a smaller number [Proc. London Math. Soc. (3) 1, 257–283, 385–414, 415–434 (1951); these Rev. 13, 627, 825]. The author remarks that (*) means that Q is equivalent to a form $\alpha x^2+\beta xy+\gamma y^2$ with

$$(**) \quad -D^2(2\sqrt{5}+2)^{-1} \leqq \alpha\gamma<0,$$

and that he has already proved that every Q is equivalent to a reduced form with not merely (**) but also

$$\frac{1}{|\alpha|}+\frac{1}{|\gamma|}>\frac{3+\sqrt{5}}{D}, \quad D-\beta<(2\sqrt{5}-4)(|\alpha|+|\gamma|);$$

with the same set of exceptions [ibid. (2) 44, 323–335 (1938)]. He further proves that Q is always equivalent to a form with $\max(|\alpha|, |\gamma|) \leqq \frac{1}{2}D$; but there are indenumerably many forms which are never equivalent to a form with $\max(|\alpha|, |\gamma|) \leqq (\frac{1}{2}-\epsilon)D$ for any preassigned $\epsilon>0$.

J. W. S. Cassels (Cambridge, England).

Cohen, Eckford. Rings of arithmetic functions. II. The number of solutions of quadratic congruences. Duke Math. J. 21, 9–28 (1954).

In part I [same J. 19, 115–129 (1952); these Rev. 13, 823] sums of the form

$$c(n,r)=\sum_{\substack{(x,r)=1\\0<x<r}}\epsilon(xn,r),\quad\text{where}\quad\epsilon(xn,r)=e^{2\pi ixn/r},$$

were considered. Here the author studies quadratic functions of the form

$$T(n,r)=\sum_{\substack{(x,r)=1\\0<x<r}}\epsilon(x^2n,r),\quad r\text{ odd},\quad T(n,1)=1,$$

and sums of the form

$$J(n,r)=\sum_{\substack{(x,r)=1\\0<x<r}}\binom{x}{r}\epsilon(nx,r).$$

Connections between the J function and the c and T functions are given. It is shown that T has multiplicative properties. It is further shown that the number $N_s(n,r)$ of representations of a number n as a sum of s squares (mod r), $n\equiv a_1x_1^2+\cdots+a_sx_s^2 \bmod r$, where r is odd and the a_i prime to r, is connected with a function $\mathfrak{S}_s(n,r)$ involving the c, J, and the ordinary Gauss sums $S(n,r)$. *H. Bergström.*

Segre, Beniamino. Generalizzazione di un'identità aritmetica di Conforto. Archimede 6, 100–106 (1954).

The general identities considered are

$$\sum_{h=1}^{r}\{2\cos(2h\pi/r)\}^n=r\cdot\sum_{2h\equiv n(\mathrm{mod}\ r)}\binom{n}{h}$$

and

$$(-1)^{[n/2]}\sum_{h=1}^{r}\{2\sin(2h\pi/r)\}^n=r\cdot\sum_{2h\equiv n(\mathrm{mod}\ r)}(-1)^h\binom{n}{h},$$

and they are derived by a simple combination of the binomial theorem and cyclotomy. Special cases are considered in which the left members are simplified to give formulas for sums of the binomial coefficients involved. The special cases are $r=8$ (n even), $r=3$, and $r=5$, the last case involving the Fibonacci numbers. The author concludes with the comment that left members are integral multiples of r which give known congruence relations. *D. H. Lehmer.*

Vandiver, H. S. The relation of some data obtained from rapid computing machines to the theory of cyclotomic fields. Proc. Nat. Acad. Sci. U. S. A. 40, 474–480 (1954).

This paper discusses the implications of results recently obtained by the SWAC on cyclotomic fields [same Proc. 40, 25–33 (1954); these Rev. 15, 778]. These results, as they concern the class number in particular, lead to what the author describes as an entirely new point of view. Thus far the so-called second factor of the class number has not been divisible by the prime defining the cyclotomic field. Although the number of fields which are irregular is rather larger than the previous somewhat limited evidence might have suggested, not one of the 118 irregular fields is improperly irregular. The author collects a number of results on properly irregular cyclotomic fields and discusses the nature of their proofs. The numerical results seem to indicate that there may well be an infinity of regular cyclotomic fields. *D. H. Lehmer* (Berkeley, Calif.).

Urazbaev, B. M. On an asymptotic formula in algebra. Doklady Akad. Nauk SSSR (N.S.) 95, 935–938 (1954). (Russian)

Let l be a prime number. Then the number of cyclic fields of degree l and of discriminant at most x^{l-1} is $\lambda x+O(x^{1-1/(l-1)+\epsilon})$, where λ is a constant depending only on l. Further, if all the integers of all the cyclic fields of degree l are plotted in l-dimensional space with their conjugates as co-ordinates, then the number of points in a sphere of radius r is $\nu_3r^3\log r+O(r^3)$ ($l=3$), $\nu_lr^l+O(r^{l-1})$ ($l>3$), where ν_l depends only on l. The proof of the first result depends on noting that the discriminant is always of the form $(l^\alpha p_1\cdots p_k)^{l-1}$, where $\alpha=0$ or 2, and the primes $p\equiv1\ (l)$, and that the number of fields with given discriminant is $(l-1)^k$ if $\alpha=2$ but $(l-1)^{k-1}$ if $\alpha=0$. The result now follows from evaluating the residue of

$$\frac{\phi(s)x^s}{s(s+1)}\quad\text{at } s=1,\text{ where }\quad\phi(s)=\prod_{p\equiv1\ (l)}\left(1+\frac{l-1}{p^s}\right)$$

and using standard techniques. The second result is only stated but is said to follow from a similar method and the results of an earlier paper [Izvestiya Akad. Nauk Kazah. SSR 1951, no. 62, Ser. Mat. Meh. 5, 37–52 (1951); these Rev. 15, 403]. *J. W. S. Cassels* (Cambridge, England).

Urazbaev, B. M. An asymptotic formula for the growth of the number of Abelian fields of degree l^2. Doklady Akad. Nauk SSSR (N.S.) 95, 1145–1147 (1954). (Russian)

An Abelian field over the rationals with group of type (l, l) (l prime) for which l is not critical has a discriminant of the type $D=(p_1p_2\cdots p_k)^{l(l-1)}$ for some k, where the p_j are primes $p\equiv1\ (l)$; and to each D correspond

$$\frac{(l+1)^{k-1}-1}{l}(l-1)^{k-2}$$

distinct fields. The author shows that the number of such fields with $D\leqq x^{l(l-1)}$ is of the form

$$xf(\log x)+O(x^{1-1/(l^2-1)+\epsilon}),$$

where $f(y)$ is a polynomial of degree l in y with constant coefficients (depending on l). The proof, which is only sketched, depends on evaluating the residue at $s=1$ of $\phi(s)x^s/s(s+1)$ where

$$\phi(s)=\prod_{p\equiv1\ (l)}\left(1+\frac{l^2-1}{p^s}\right),$$

and follows quite closely a standard proof of the Prime Number Theorem [cf. Ingham, The distribution of prime numbers, Cambridge, 1932]. *J. W. S. Cassels.*

Iwasawa, Kenkichi. A note on Kummer extensions. J. Math. Soc. Japan 5, 253–262 (1953).

Let K be the composite of all Kummer extensions of a field k (in an algebraic closure of k). The author determines as follows the (topologized) Galois group G of K/k. Let k^* be the multiplicative group of elements $\neq0$ in k, $W(k)$ the group of roots of unity in k and $\bar{R}(k)$ the subgroup of the rationals modulo 1 to which $W(k)$ is (noncanonically) isomorphic. Then G is canonically isomorphic to

$$\mathrm{Hom}\,(k^*,\mathrm{Hom}\,(\bar{R}(k),W(k))),$$

where k^*, $\bar{R}(k)$ and $W(k)$ are discrete and the "Hom" groups are taken with their compact topologies. This allows one to determine G when the structure of k^* is known.

It is proved that the multiplicative group of elements $\neq 0$ of an Abelian extension k of a finite algebraic number field is isomorphic to the product of the group of roots of unity in k by a free Abelian group. Assuming further that k contains all roots of unity, it follows that the Galois group of the maximal Abelian extension of k is isomorphic to the product of countably many times the character group of the rationals modulo 1. This leads to a determination of the factors of the successive commutator groups of the Galois group Γ of the maximal solvable extension of k. The structure of the group Γ itself has been determined by the author by a different method [Ann. of Math. (2) **58**, 548–572 (1953); these Rev. **15**, 509]. *C. Chevalley.*

Lapin, A. I. Theory of Šafarevič's symbol. Izvestiya Akad. Nauk SSSR. Ser. Mat. **17**, 31–50 (1953). (Russian)

Ce travail est consacré à la démonstration, sous l'hypothèse que le corps k, dont il s'agit, est p-adique (au sens restreint), de l'invariance (c'est-à-dire de l'indépendance du choix de π) du symbole $(\lambda, \mu)_\pi$ de Chafarévitch (Šafarevič), défini par cet auteur dans son travail sur la loi de réciprocité explicite [Mat. Sbornik N.S. **26**(68), 113–146 (1950); ces Rev. **11**, 230], en se plaçant dans le cas général des exposants primaires p^n (dans le cas $n=1$ des exposants premiers, cette invariance a été déjà prouvée par Chafarévitch lui-même dans son travail cité). Contrairement aux autres démonstrations de cette invariance, faites indépendamment par Hasse [Math. Nachr. **5**, 301–327 (1951); ces Rev. **13**, 113] et Kneser [ibid. **6**, 89–96 (1951); ces Rev. **13**, 726], la démonstration considérée est purement locale et ne présuppose pas la connaissance préalable de la théorie locale ou globale de la théorie des corps de classes ou de la théorie du symbole des restes normiques. L'auteur prouve l'invariance de $(\lambda, \mu)_\pi$ en démontrant sa "propriété normique": $\lambda \in k$ est une norme de $k(\mu^{1/p^n})$ à k $(\mu \in k)$ si, et seulement si, $(\lambda, \mu)_\pi = 1$. La méthode est essentiellement celle du travail cité de Chafarévitch, mais avec l'emploi fréquent du lemme suivant de la Thèse de Chevalley: si un élément α de k est à la fois une norme de $k(\lambda^{1/m})$ à k et de $k(\mu^{1/m})$ à k, il l'est aussi de $k((\lambda\mu)^{1/m})$ à k $(\lambda, \mu \in k)$. La démonstration considérée est trop technique pour pouvoir en donner une idée plus détaillée.

A la fin de ce travail, une construction de la théorie locale des corps de classes à partir des propriétés du symbole $(\lambda, \mu) = (\lambda, \mu)_\pi$ est esquissée en se bornant au cas, où le corps de base contient les racines p^n-ièmes de l'unité et où l'exposant du groupe de Galois des extensions abéliennes considérées divise p^n.

M. Krasner (Paris).

Lapin, A. I. On the theory of Šafarevič's symbol. Izvestiya Akad. Nauk SSSR. Ser. Mat. **18**, 145–158 (1954). (Russian)

L'auteur démontre l'invariance du symbole $(\ ,\ )$ de Chafarévitch dans le cas d'exposant premier p. Il le fait plus simplement que cela n'a été fait dans le travail original de Chafarévitch [Mat. Sbornik N.S. **26**(68), 113–146 (1950); ces Rev. **11**, 230] et dans son propre travail analysé ci-dessus, consacré au cas général des exposants primaires. La théorie locale des corps de classes, pour le corps de base contenant les racines p-ièmes de l'unité et pour les extensions abéliennes dont le groupe de Galois est de type $(p, p, \cdots, p)$, est construite d'une manière plus détaillée que dans le travail cité de l'auteur.

M. Krasner (Paris).

Faddeev, D. K. On a hypothesis of Hasse. Doklady Akad. Nauk SSSR (N.S.) **94**, 1013–1016 (1954). (Russian)

Soit k/k_0 une extension galoisienne finie, dont Γ soit le groupe de Galois. G étant un groupe donné d'avance, dont Γ est une image homomorphe, et φ étant un homomorphisme prescrit de G sur Γ, il se pose la question de la possibilité de l'immersion de k/k_0 dans une algèbre galoisienne (au sens de Hasse) L/k de groupe G, cette immersion étant telle que l'application canonique de G sur le groupe de Galois de k/k_0 coïncide avec φ. L'auteur avait donné, dans un travail antérieur [Mat. Sbornik N.S. **15**(57), 243–284 (1944); ces Rev. **6**, 200], ensemble avec B. Delaunay, la condition nécessaire que voici de cette immersibilité, appelée "condition de cohérence": g étant le noyau de φ, considérons G comme un groupe d'opérateurs de l'anneau de groupe $g \cdot k$ de g sur k, en posant, pour tout $\sigma \in G$, $\sigma \cdot a = a^{\varphi(\sigma)}$ si $a \in k$ et en posant $\sigma \cdot \tau = \sigma^{-1}\tau\sigma$ quand $\tau \in g$. Alors la condition de cohérence consiste en l'existence de 1-cochaine de G à coefficients dans le demigroupe multiplicatif de $g \cdot k$, qui soit un cocycle et dont la restriction à g soit x^{-1}.

Hasse, dans un cas plus général, a également donné une condition nécessaire d'une telle immersibilité, qui se trouve être équivalente, dans le cas considéré, à la condition précédente. Il a émis l'hypothèse que cette condition nécessaire d'immersibilité est aussi suffisante. L'auteur montre, par un contre-exemple, qui ne fait intervenir que les composés des extensions quadratiques, qu'il n'en est rien.

M. Krasner (Paris).

Kuniyoshi, Hideo, and Takahashi, Shuichi. On the principal genus theorem. Tôhoku Math. J. (2) **5**, 128–131 (1953).

A generalization of the principal-genus theorem to arbitrary finite Galois extensions K/k of an algebraic number field k is given which is similar to that given by Terada for the case of abelian extensions [same J. (2) **4**, 141–152 (1952); these Rev. **14**, 729]. In terms of ideles (instead of ideals), this result is equivalent (as is shown in the paper) to the following. Let D be the different of K relative to k, and let f be the 'conductor' which is here defined by making its local component for a prime P of K equal to P^e, where e is the sum of the orders of all the non-trivial ramification groups for P relative to k. Let J be the idele group of K and, for any divisor S of K, denote by $J(S)$ the subgroup of J consisting of the ideles whose P-components are units when P does not occur in S, and are congruent to 1 modulo the contribution of P to S otherwise. The result can now be stated as follows: Let c be a 1-cochain for the Galois group G of K/k in J whose coboundary is the product of a 2-cocycle for G in the multiplicative group of K by a 2-cochain for G in $J(fD)$. Then c is the product of a coboundary for G in J by a 1-cochain for G in $J(f)$.

This result is derived from a lemma which says that the canonical image of the 2-dimensional cohomology group for G in $J(fD)$ is trivial. This, in turn, is derived from the corresponding local result which is established by using the fact that the cohomology groups of a Galois group in the additive group of the field are trivial. *G. Hochschild.*

Terada, Fumiyuki. A note on the principal genus theorem. Tôhoku Math. J. (2) **5**, 211–213 (1953).

It is shown that a weakened form of the author's earlier generalization of the principal-genus theorem [same J. (2) **4**, 141–152 (1952); these Rev. **14**, 729] can be derived from

the theorem of Kuniyoshi and Takahashi quoted in the preceding review. *G. Hochschild* (Urbana, Ill.).

Furtado Gomide, Elza. On the theorem of Artin-Weil. Bol. Soc. Mat. São Paulo 4 (1949), 1–18 (1951). (Portuguese)

Consider the equation

$$(*) \qquad \sum_{i=1}^{r} a_i x_1^{m_{1i}} \cdots x_s^{m_{si}} = 0 \quad (a_i \in \mathrm{GF}(q)),$$

where the m_{ji} are arbitrary positive integers. Let $\bar{N}_\nu$ denote the number of non-zero solutions of (*) with $x_i \in \mathrm{GF}(q^\nu)$. It is proved that

$$\bar{N}_\nu = q^{-r}(q-1)^s - (-1)^r(q^\nu-1)^{s-r+1} + (q^\nu-1)^{s-r+1}\sum_{\alpha} \chi_{\alpha_1}(a_1)\cdots\chi_{\alpha_r}(a_r) j^*(\alpha),$$

where the χ's are characters of the multiplicative group of $\mathrm{GF}(q^\nu)$, $j(\alpha)$ is a Jacobi sum, and the summation is over all rational α_i such that

$$(q^\nu-1)\alpha_i \equiv 0, \quad \sum_{i=1}^{r} \alpha_i \equiv 0 \pmod 1,$$

$$\sum_{i=1}^{r} m_{ji}\alpha_i \equiv 0 \pmod 1 \quad (j=1, \cdots, s).$$

Let the matrix (m_{ji}) be of rank r. If N_ν denotes the number of rational points over $\mathrm{GF}(q^\nu)$ that satisfy (*), then it is shown that

$$\sum_{\nu=1}^{\infty} N_\nu Z^{\nu-1} = \frac{d}{dZ} \log R(Z),$$

where $R(Z)$ is a rational function of Z, thus verifying a conjecture of Weil [Bull. Amer. Math. Soc. 55, 497–508 (1949); these Rev. 10, 592] in the special case (*). *L. Carlitz.*

Schneider, Theodor. Zur Charakterisierung algebraischer Funktionen mit Hilfe des Eisensteinschen Satzes. Math. Z. 60, 98–108 (1954).

The author proves the following converse to Eisenstein's theorem: "Let $f(z)$ be an analytic function regular for $|z|<\rho$; let s and γ be given positive integers; and let $\nu_0 = 2[\rho^{-1}+1]$. If $|z_0|<\rho$, denote by

$$f(z) = \sum_{\kappa=0}^{\infty} a_\kappa (z_0)(z-z_0)^\kappa$$

the Taylor series of $f(z)$ in the neighborhood of z_0. Assume that, for

$$n \geq 6(s+1)\gamma(\nu_0+1)^2, \quad \tau = n+1, n+2, \cdots, 2n,$$

(i) the algebraic field $\mathfrak{K}_{\nu_0+\tau}$ generated by $a_0(1/(\nu_0+\tau))$ is at most of degree s; (ii) this field contains also all the coefficients $a_\kappa = a_\kappa(1/(\nu_0+\tau))$ where $\kappa = 1, 2, \cdots$; (iii) there exists a positive integer $T = T(1/(\nu_0+\tau))$ such that the products $T^{\kappa+1}a_\kappa$ ($\kappa = 0, 1, 2, \cdots; \tau = n+1, n+2, \cdots, 2n$) are algebraic integers; (iv) for all these κ and τ

$$\max\,(T^{\kappa+1}, |\overline{T^{\kappa+1}a_\kappa}|) < (\nu_0+\tau)^{\gamma(\kappa+1)}.$$

Then $f(z)$ is a branch of an algebraic function." If, in particular, $\mathfrak{K}_{\nu_0+\tau}$ is independent of τ and if n satisfies the stronger inequality

$$n \geq 10^4(s+1)^3\gamma^2(\nu_0+1)^2,$$

then $f(z)$ is a rational function of z. For the proof, polynomials

$$\Phi(z) = \sum_{\lambda=0}^{l}\sum_{\mu=0}^{m} c_{\lambda\mu} z^\lambda f(z)^\mu$$

with rational integral coefficients $c_{\lambda\mu}$ not all zero are constructed that vanish to a high order at the places $1/(\nu_0+\tau)$, and it is shown that these vanish there identically. The paper is unfortunately not free of misprints; thus, in the statement of the theorem, the condition for τ is wrongly given as $\tau = 1, 2, \cdots, n$. *K. Mahler* (Manchester).

Bateman, P. T., and Chowla, S. The equivalence of two conjectures in the theory of numbers. J. Indian Math. Soc. (N.S.) 17 (1953), 177–181 (1954).

Let $\lambda(n)$ be Liouville's function (i.e. the purely multiplicative function for which $\lambda(p) = -1$ for p a prime). Define $L_1(x) = \sum_{\nu \leq x} \lambda(\nu)\nu^{-1}$, and $\xi(\chi, x) = \sum_{\nu \leq x} \chi(\nu)\nu^{-1}$, where χ is a real character function. Turán conjectured [Danske Vid. Selsk. Mat.-Fys. Medd. 24, no. 17 (1948); these Rev. 10, 286] that $L_1(x) > 0$ for every x, and observed that this implies the Riemann hypothesis. The SWAC has shown that $L_1(x) > 0$ for $x < 100000$. The second conjecture,

$$\xi(\chi, x) > 0 \text{ for every } x \text{ and every } \chi,$$

is shown by the authors to be equivalent to the first. The only difficulty is in showing that the first implies the second. This theorem, communicated to the authors by A. B. Showalter, is proved by double induction for a more general class of functions χ satisfying the following conditions: $\chi(mn) = \chi(m)\chi(n)$, $\chi(1) = 1$, $\chi(p) = -1$ or $\chi(p) \geq 0$ (p a prime). The authors point out that for a fixed character $\chi \pmod D$, say $\xi(\chi, x) > 0$ for all sufficiently large x since $\xi(\chi, \infty)$ is proportional to the class number of D. It is shown that (*) $\xi(\chi, x) > 0$ for all $x > 1.75D \log D$ and that, provided $D > D_0$, (*) holds for all $x > D^{1+\epsilon}$. *D. H. Lehmer.*

Müller, Claus. Eine Verallgemeinerung der Eulerschen Summenformel und ihre Anwendung auf Fragen der analytischen Zahlentheorie. Abh. Math. Sem. Univ. Hamburg 19, no. 1–2, 41–62 (1954).

Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be two linearly independent plane vectors. Put $\mathfrak{g}_i\mathfrak{g}_j = g_{ij}$, $g = \det(g_{ij})$, $g^{ij}g_{jk} = \delta_k^i$, and define $\mathfrak{g}^i$ by means of $\mathfrak{g}^i = g^{ij}\mathfrak{g}_j$ so that $\mathfrak{g}^i\mathfrak{g}_k = \delta_k^i$. The totality of vectors $\mathfrak{g} = n^1\mathfrak{g}_1 + n^2\mathfrak{g}_2$, where the n's are rational integers, define a modul (or lattice) Ω, the vectors $\mathfrak{h} = n_1\mathfrak{g}^1 + n_2\mathfrak{g}^2$ define the reciprocal lattice Ω^{-1}. The fundamental domain of Ω is denoted by $\mathfrak{F}$. If $\mathfrak{x}$ is an arbitrary vector we put $\mathfrak{x} = x^1\mathfrak{g}_1 + x^2\mathfrak{g}_2$ and define the operator $\Delta = (\partial/\partial x^1)^2 + (\partial/\partial x^2)^2$. Also write $e(x) = e^{2\pi i x}$, $\phi(\mathfrak{hx}) = g^{-1/4}e(\mathfrak{hx})$.

A fundamental function $G(\lambda; \mathfrak{x})$ is defined by means of the following properties: (1) $G(\lambda; \mathfrak{x}+\mathfrak{g}) = G(\lambda; \mathfrak{x})$ for $\mathfrak{x} \neq \mathfrak{g}$ and all $\mathfrak{g} \in \Omega$; (2) $G(\lambda; \mathfrak{x})$ is analytic for $x \neq g$; moreover,

$$\Delta G(\lambda; \mathfrak{x}) + G(\lambda; \mathfrak{x}) = g^{-1/2} \sum_{\lambda = 4\pi^2\mathfrak{h}^2} e(\mathfrak{hx});$$

(3) $G(\lambda; \mathfrak{x}) + (2\pi)^{-1}\log|\mathfrak{x}|$ is continuously differentiable in $\mathfrak{F}$; (4) if $\lambda = 4\pi^2\mathfrak{h}^2$, then $\int_{\mathfrak{F}} G(\lambda; \mathfrak{x})\overline{\phi(\mathfrak{hx})}\,dF_{\mathfrak{x}} = 0$. It is proved that $G(\lambda; \mathfrak{x})$ is uniquely determined by these properties. Also we have the expansion

$$G(\lambda; \mathfrak{x}) \sim g^{-1/2} \sum_{4\pi^2\mathfrak{h}^2 \neq \lambda} \frac{e(\mathfrak{hx})}{4\pi^2\mathfrak{h}^2 - \lambda}.$$

For $\lambda = 0$, put $G(\mathfrak{x}) = G(0; \mathfrak{x})$ and let

$$a_0 = a_0(\Omega) = \lim_{\mathfrak{x}=0} (G(\mathfrak{x}) + (1/2\pi)\log|\mathfrak{x}|);$$

also put $\Theta(\tau)=\prod_{n=1}^{\infty}(1-e^{2\pi i\tau})$, $\tau=(g_{12}+ig^{1/2})/g^{22}$. It is shown that

$$a_0(\Omega^{-1})=a_0(\Omega)-\frac{1}{4\pi}\log g,$$

$$a_0(\Omega)=-\frac{1}{2\pi}\log\frac{2\pi}{\sqrt{g_{22}}}|\Theta(\tau)e^{\pi i\tau/12}|^2.$$

If now $\mathfrak{G}$ is a regular domain with continuously differentiable boundary curve $\mathfrak{E}$ and $f(\mathfrak{x})$ has continuous second derivatives in $\mathfrak{G}$, then the author's extension of the Euler-MacLaurin summation formula is contained in the following:

$$(*)\quad \sum_{\mathfrak{g}\in\mathfrak{G}}{}'f(\mathfrak{g})=g^{-1/2}\int_{\mathfrak{G}}f(\mathfrak{x})dF-\int_{\mathfrak{G}}G(\mathfrak{x})\Delta f\,dF$$

$$-\int_{\mathfrak{E}}\left(f\frac{\partial}{\partial n}G(\mathfrak{x})-G(\mathfrak{x})\frac{\partial}{\partial n}f\right)ds,$$

where $\sum'_{\mathfrak{g}\in\mathfrak{G}}f(\mathfrak{g})=\sum_{\mathfrak{g}\in\mathfrak{G}}f(\mathfrak{g})-\frac{1}{2}\sum_{\mathfrak{g}\in\mathfrak{G}}f(\mathfrak{g})$ and $\partial/\partial n$ denotes the derivative in the direction of the exterior normal.

Application of (*) is made first to the Epstein zeta-function $\zeta(s,\Omega)=\sum_{\mathfrak{g}\neq 0}|g|^{-s}$ (Re $(s)>2$). Proof is given of the analytic continuation and the functional equation of $\zeta(s,\Omega)$, also of the Kronecker formula

$$\lim_{s=2}\left(\zeta(s,\Omega)-\frac{2\pi}{\sqrt{g}}\frac{1}{s-2}\right)=a_0(\Omega).$$

As a second application the author derives the Hardy-Landau identity

$$\sum_{|\mathfrak{g}|\leq R}{}'1=\frac{\pi R^2}{\sqrt{g}}+R\sum_{\substack{\mathfrak{h}\in\Omega^{-1}\\ \mathfrak{h}\neq 0}}\frac{J_1(2\pi R|\mathfrak{h}|)}{|\mathfrak{h}|}.$$

L. Carlitz (Durham, N. C.).

Müller, Claus. Eine Formel der analytischen Zahlentheorie. Abh. Math. Sem. Univ. Hamburg **19**, no. 1–2, 62–65 (1954).

As an application of his generalization of the Euler-MacLaurin summation formula (see the preceding review), the author proves the following formula:

$$\prod_{n=1}^{\infty}\left(\frac{n}{N}\right)^{r(n)}=\frac{\Gamma^4(\frac{1}{4})}{4\pi}Ne^{-\pi N}\{1+O(N^{-1/4})\},$$

where $r(n)$ denotes the number of representations of n as a sum of two squares. *L. Carlitz* (Durham, N. C.).

Müller, Claus. Eine Erweiterung der Hardyschen Identität. Abh. Math. Sem. Univ. Hamburg **19**, no. 1–2, 66–76 (1954).

The author first extends his generalization of the Euler-MacLaurin summation formula (see the second preceding review) and then proves the following generalization of the Hardy-Landau identity:

$$\lim_{n\to\infty}R\sum_{|\mathfrak{g}-\mathfrak{n}|\leq N}e^{2\pi i(\mathfrak{a}\mathfrak{g})}\frac{J_1(2\pi R|\mathfrak{g}-\mathfrak{n}|)}{|\mathfrak{g}-\mathfrak{n}|}$$

$$=g^{-1/2}e^{2\pi i(\mathfrak{a}\mathfrak{n})}\sum_{|\mathfrak{h}+\mathfrak{a}|\leq R}{}'e^{2\pi i(\mathfrak{h}\mathfrak{n})}.$$

Here $\mathfrak{g}\in\Omega$, $\mathfrak{h}\in\Omega^{-1}$, the lattice reciprocal to Ω, and

$$\sum_{|\mathfrak{h}+\mathfrak{a}|\leq R}{}'=\sum_{|\mathfrak{h}+\mathfrak{a}|\leq R}-\frac{1}{2}\sum_{|\mathfrak{h}+\mathfrak{a}|=R}.$$

L. Carlitz (Durham, N. C.).

Bredihin, B. M. On sum functions of characters of numerical semigroups. Doklady Akad. Nauk SSSR (N.S.) **94**, 609–612 (1954). (Russian)

Suppose G is a multiplicative group of positive real numbers with a finite basis $\omega_1, \omega_2, \cdots, \omega_N$, where it is assumed that $\omega_k>1$ for $k=1, 2, \cdots, N$, and let S be the semigroup generated by $\omega_1, \omega_2, \cdots, \omega_N$. The author proves a conjecture of A. O. Gel'fond to the effect that if χ is any (bounded) character of G and $H(x)=\sum_{\alpha\in S,\,\alpha\leq x}\chi(\alpha)$, then $H(x)$ is unbounded. More specifically, he proves that if $M_{k,l}(x)$ is the number of convergents of the continued fraction for $\log\omega_l/\log\omega_k$ $(k\neq l)$ which have denominators in the interval $[0, x]$, then

$$\max_{t\in[0,x]}|H(t)|\geq c[\max_{k\neq l}M_{k,l}(x^{1/2}\log^{-3/2}x)]^{1/2},$$

where c is a suitable positive number. The proof belongs essentially to the domain of analytic number theory and uses ideas and methods from an earlier note of the author [same Doklady (N.S.) **90**, 707–710 (1953); these Rev. **15**, 105] and a paper of Čudakov and Pavlyučuk [Trudy Mat. Inst. Steklov. **38**, 366–381 (1951); these Rev. **15**, 105].

P. T. Bateman (Urbana, Ill.).

Herrmann, Oskar. Über Hilbertsche Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung. Math. Ann. **127**, 357–400 (1954).

In his thesis [Free Univ. of Amsterdam, 1943; these Rev. **8**, 8] the reviewer gave a first approach to a theory of Hecke's T-operators for Hilbert's modular functions. The present author goes much further and obtains the generalization of the entire Hecke-Petersson theory. The main features are: (1) The author constructs a fundamental domain for the Hilbert group of modular substitutions in a totally real field K, and more generally for the groups $\Gamma_1(K_a)$ (see below). These domains have cusp points (h is the number of ideal classes of K). He shows that the linear family of modular forms of a given dimension has a finite basis. Further, applying Petersson's metrization method [Math. Ann. **116**, 401–412 (1939)] he shows that the linear family of cusp forms is spanned by forms which are eigenfunctions of all T-operators. Finally, he constructs, by Mellin's transformation, the corresponding Dirichlet series. The eigenfunctions of the T-operators correspond to Dirichlet series with an Euler product expansion. (2) The reviewer was able to define suitable operators $T(\mathfrak{n})$ only if $\mathfrak{n}$ is equivalent to an ideal square, so that a closed theory was to be expected only in the case where h is odd. The present author solves this difficulty by replacing modular forms by vectors of h different forms, belonging to h different transformation groups $\Gamma_1(K_a)$ $(a=1, \cdots, h)$, which do not seem to have been studied before. These groups are described in terms of ideal numbers: The set of numbers $\neq 0$ of K is extended to a multiplicative system Z^+ of ideal numbers, corresponding to ideals of K, in such a way that no new units are involved [see Hecke, Math. Z. **6**, 11–51 (1920)]. K_a is the subset corresponding to the ideals of class C_a ($C_1, \cdots, C_h$ are the ideal classes of K; C_1 is the unit ideal class, and $K_1=K$). Now $\Gamma_1(K_a)$ is defined as the set of all matrices $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$ where $\alpha\delta-\beta\gamma=1$, α, β, γ, δ integers, $\alpha\in K$, $\delta\in K$, $\gamma\in K_a$, $\beta\in(K_a)^{-1}$. The fact that $K_1, \cdots, K_h$ need not be totally real is irrelevant. $\Gamma_1(K_1)$ is the Hilbert group.

The use of ideal numbers has the advantage that no non-integral ideals or auxiliary ideals are needed. On the other

hand it has the disadvantage that the ranges of the complex variables τ can be entirely different, and that the relations between the several groups are not easily noticed. Some of these, not mentioned by the author, are (i) $\Gamma_1(K_\mathfrak{a})$ is isomorphic (similarity) with a group $\Gamma(\mathfrak{a})$ whose intersection with the Hilbert group Γ_1 has the same finite index both in $\Gamma(\mathfrak{a})$ and in Γ_1. Here $\mathfrak{a}$ is an integral ideal of class $C_\mathfrak{a}$, and $\Gamma(\mathfrak{a})$ consists of all matrices $\begin{pmatrix}\lambda & \mu\\ \nu & \rho\end{pmatrix}$ with $\lambda,\mu,\nu,\rho \in K$, $\lambda\rho-\mu\nu=1$, $\lambda \in (1)$, $\rho \in (1)$, $\mu \in \mathfrak{a}$, $\nu \in \mathfrak{a}^{-1}$. Therefore, $\Gamma(\mathfrak{a})$ contains the Hilbert group of level ("Stufe") $\mathfrak{a}$, and so the author's modular forms are trivially related to special modular forms of level $\mathfrak{a}$ in the sense of Kloosterman [Abh. Math. Sem. Univ. Hamburg. 6, 163–188 (1928)]. (ii) There is also an isomorphism $\Gamma(\mathfrak{a})\simeq\Gamma(\mathfrak{b})$, if $\mathfrak{a}\mathfrak{b}^{-1}$ is equivalent to the square of an ideal. This fact furnishes the connection between the author's results and those of the reviewer, whose T-operators were defined in a different way.

Misprints: In the definition of the T-operator on page 379 $R(\eta, \kappa, \sigma, K)$ should be $R(\eta, \kappa, \sigma, K_\mathfrak{a})$. In the formula for $D(s, \lambda)$ on page 397 the factor $N(\sigma)^{-s}$ should be added on the right. *N. G. de Bruijn* (Amsterdam).

Lekkerkerker, C. G. The elements of the theory of modular forms, derived from properties of theta series. Math. Centrum Amsterdam. Rapport ZW 1953-006, 17 pp. (1953).

Expository paper.

*Vinogradov, I. M. **The method of trigonometrical sums in the theory of numbers.** Translated, revised and annotated by K. F. Roth and Anne Davenport. Interscience Publishers, London and New York. x+180 pp. $5.00.

A translation, with expansions and modifications of the text as well as annotations, of the author's monograph in Trudy Mat. Inst. Steklov. 23 (1947) [these Rev. 10, 599].

Rogers, C. A. A note on the theorem of Macbeath. J. London Math. Soc. 29, 133–143 (1954).

A point x of a set G in n-dimensional space is called exterior if there is a half space H such that $H\cap G=x$. Let Γ be a grid of determinant $d(\Gamma)$ and R be an open convex set not an infinite cylinder. If $\Gamma\cap R$ is not empty it has an exterior point and every exterior point x satisfies (1) $\Delta(S(R, x))\leqq d(\Gamma)$, where $(R, x)=\{R-x\}\cap\{-R+x\}$ and $\Delta(S)$ is the critical determinant of S. This generalizes a result of Macbeath [Ann. of Math. (2) 56, 269–293 (1952); these Rev. 14, 624]. The proof is simple. Conditions when an infinitude of points of $\Gamma\cap R$ satisfy (1) are given. Let $V(R, x)$ be the volume of (R, x) and $V(R, \Gamma)$ the g.l.b. of $V(R, x)$ for $x \in \Gamma\cap R$. There exists a constant $\lambda_n<1$ and dependent only on the dimension n such that if $x \in \Gamma\cap R$ and $V(R, x)<V(R, \Gamma)/\lambda_n$, then x is an exterior point of $\Gamma\cap R$. *L. Tornheim* (Ann Arbor, Mich.).

Lursmanašvili, A. P. On the number of lattice points in multidimensional spheres of odd dimension. Soobščeniya Akad. Nauk Gruzin. SSR 14, 513–520 (1953). (Russian)

Let $A_k(x)$ be the number of lattice points in the k-dimensional sphere $z_1^2+z_2^2+\cdots+z_k^2\leqq x$. Write

$$A_k(x)=V_k(x)+P_k(x),$$

where $V_k(x)$ is the content of the sphere. The author obtains the result

$$P_{2k+1}(x)=\pi^{k+\frac{1}{2}}\{Z(k)\}^{-1}\sum_{s=1}^{m}\Phi_{k,s}(x)x^{k-s+\frac{1}{2}}+O(x^{k-m}),$$

where $Z(k)=(1-2^{-k})\zeta(k)$ for even k, and $\sum_{n=1}^{\infty}(-1/n)n^{-k}$ for odd k. Also

$$\Phi_{k,s}(x)=\sum_{n=1}^{\infty}\left(\frac{-1}{n}\right)^k n^{s-k-1}\vartheta_s(x, n)+\frac{1}{2}\sum_{n=1}^{\infty}(2n)^{s-k-1}G_s(x, n).$$

Here $G_s(x, n)$ is a linear combination of $\vartheta_s(x, l)$ for $l=n$, $2n$, $4n$, and $\vartheta_s(x-n, 4)$ and

$$\vartheta_s(x, l)=\sum_{h=1}^{l}\varphi_s\left(\frac{x-h^2}{l}\right),$$

where $\varphi_s(z)$ is a multiple of the sth Bernoullian polynomial (continued as a periodic function outside $0\leqq z<1$). This result is obtained from a similar result for $P_{2k}(x)$ obtained by the author in an earlier paper [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 19, 79–120 (1953)]. *R. A. Rankin* (Birmingham).

ANALYSIS

*Rey Pastor, J. **Elementos de la teoria de funciones.** [**Elements of the theory of functions.**] 3rd ed. Ibero-Americana, Madrid-Buenos Aires, 1953. 560 pp.

This volume combines a rigorous course in calculus, the fundamentals of complex-variable theory, a rather detailed introduction to real-variable theory (to and beyond Lebesgue integration), and an introduction to Fourier series. The style is rather more condensed than is customary in textbooks in English, but the author has evidently devoted a great deal of thought to the presentation of his material. He is a master of the art of intelligible condensation of a proof and is consequently able to present many topics which are frequently omitted in a course of this kind. The exposition follows fairly traditional lines, but with many improvements in arrangement and details. Among a number of details which attracted the reviewer's attention are an especially transparent example of two functions, not differing by a constant, but having the same derivatives everywhere (infinite values allowed) [p. 142], and a neat approach to the approximation of a continuous function by polynomials [p. 414]. The last chapter deals in some detail with the author's R^ϱ integral, defined by means of Riemann upper and lower sums using partitions into open, closed and zero-measure sets (instead of into intervals). The absolute R^ϱ integral coincides with the Lebesgue integral, but the conditional R^ϱ integral [whose definition does not seem sufficiently detailed] has a wider scope.

R. P. Boas, Jr. (Evanston, Ill.).

Vota, Laura. Medie integrali. Univ. e Politecnico Torino. Rend. Sem. Mat. 12, 283–292 (1953).

For a given continuous and strictly monotonic function $x=f(t)$, the author defines an integral mean-value function of two variables by $M(x, x)=x$ and

$$M(x_1, x_2)=\frac{1}{t_2-t_1}\int_{t_1}^{t_2}f(t)dt,$$

where $t_j=f^{-1}(x_j)$, $j=1, 2$, $x_1\neq x_2$. Properties of these

means are discussed, and functions $f(t)$ yielding the arithmetic, geometric, and harmonic means of two values are presented. *E. F. Beckenbach* (Los Angeles, Calif.).

Livingston, Arthur E. A generalization of an inequality due to Beurling. Pacific J. Math. 4, 251–257 (1954).

The (unpublished) inequality mentioned in the title is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_m b_n/\log(m+n) \leqq K\left(\sum_{m=1}^{\infty} m a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n a_n^2\right)^{\frac{1}{2}} \qquad (0<K<4\epsilon)$$

if the sums on the right hand side are finite and $a_m \geqq 0$, $b_n \geqq 0$. The author proves the following consequence of the inequalities of Hölder und Hilbert [Hardy, Littlewood, and Pólya, Inequalities, 2nd ed., Cambridge, 1952, pp. 11, 226–231; these Rev. 13, 727]:

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{\alpha(x+y)}\,d\alpha(x)\,d\alpha(y) \leqq K(\alpha)\left[\int_0^{\infty} f(x)^p\,d\alpha(x)\right]^{1/p}\left[\int_0^{\infty} g(y)^q\,d\alpha(y)\right]^{1/q}$$

[$0<K(\alpha)\leqq p+q=pq$, $p>1$, $f(x)$, $g(x)$, $\alpha(x)$ are non-negative, $\alpha(x)$ also nondecreasing and continuous from the right, $0\leqq x\leqq\infty$] if the integrals on the right exist. "The author wishes to acknowledge that any novelty in the subject matter of this note is due entirely to Professor Beurling, who suggested the . . . theorem . . . ." Some corollaries and the generalization to several variables are also discussed. *J. Aczél* (Debrecen).

Georgiev, G. Formules de quadrature mécanique pour les intégrales doubles appliquées aux polynomes du second degré. C. R. Acad. Bulgare Sci. 5 (1952), no. 1, 1–4 (1953). (Russian. French summary)

The author investigates the question of when a quadrature formula of the form

$$\iint_R \phi(x,y)\,dx\,dy = \sum_{i=1}^{3} \lambda_i \phi(x_i, y_i)$$

can hold for a given plane region R and all polynomials ϕ of degree at most 2. He obtains necessary and sufficient conditions on the three points (x_i, y_i) (which can be chosen in infinitely many ways); the λ_i are determined by the three points and are positive; they can always be made equal by proper choice of the points. *R. P. Boas, Jr.*

Castoldi, Luigi. Una rappresentazione della funzione di Dirac e il teorema di approssimazione di Weierstrass. Atti Accad. Ligure 9 (1952), 314–318 (1953).

Malliavin, Paul. Problèmes de Watson, zéros des fonctions méromorphes, équivalence de divers problèmes d'unicité. C. R. Acad. Sci. Paris 238, 2481–2483 (1954).

The author states a number of theorems on the subjects indicated in the title and on weighted polynomial approximation on a half-line, generalized quasi-analyticity, growth of functions having an asymptotic Dirichlet series, and connections among these. His point of departure is a generalized Watson problem instead of Mandelbrojt's principle of "séries adhérentes." *R. P. Boas, Jr.*

Corominas, E., and Sunyer Balaguer, F. Conditions for an infinitely differentiable function to be a polynomial. Revista Mat. Hisp.-Amer. (4) 14, 26–43 (1954). (Spanish)

The main results of this paper were announced in C. R. Acad. Sci. Paris 238, 558–559 (1954); these Rev. 15, 512. The simplest is that if $f(x) \in C^\infty$ and if to each x there is an $n(x)$ such that $f^{(n(x))}(x)=0$, then $f(x)$ is a polynomial. Proof (condensed): Let P be the complement of the set of points x such that $f(x)$ is a polynomial in some neighborhood of x. Then P is perfect. If it is not empty, the subsets of points x such that $f^{(n)}(x)=0$ exhaust P and so (Baire's theorem) one of them includes all the part Q of P in some interval I. Then $f^{(m)}(x)=0$ on Q for all $m>n$, in particular at all endpoints of complementary intervals of Q, whence $f(x)$ is a polynomial of degree less than n throughout I, a contradiction. Besides the generalizations mentioned in the review of the cited note, the authors prove that if $\{\phi_k(x)\}$ is a countable set of members of a fixed quasi-analytic class, and if, for each x, $f^{(n(x))}(x)=\phi_{\lambda(x)}^{(n(x))}(x)$, then, for some k, $f(x)-\phi_k(x)$ is a polynomial. *R. P. Boas, Jr.*

Nelšuler, L. Ya. Equations with four separating variables and their optimal two-term tabulation. Doklady Akad. Nauk SSSR (N.S.) 95, 709–712 (1954). (Russian)

The problem considered is that of finding necessary and sufficient conditions on a function $f(x_1, x_2, x_3, x_4)$ of four variables that $f=0$ be separable in the form

$$f_1(x_i, x_j) = f_2(x_p, x_q)$$

for some permutation (i, j, p, q) of $(1, 2, 3, 4)$. The problem was considered by Goursat [Bull. Soc. Math. France 27, 1–34 (1899)]. A simple solution is given by the author in the form that

$$\frac{\partial f}{\partial x_i}\Big/\frac{\partial f}{\partial x_j} \quad \text{and} \quad \frac{\partial f}{\partial x_p}\Big/\frac{\partial f}{\partial x_q}$$

be functions of (x_i, x_j) and (x_p, x_q) respectively. An equivalent condition is that three functions $\phi_1(u, v)$, $\phi_2(u, v)$, $\phi_3(u, v)$ exist such that

$$f(x_1, x_2, x_3, x_4) = \phi_3[\phi_1(x_i, x_j), \phi_2(x_p, x_q)],$$

so that the problem is connected with the tabulation of function of 4 variables via three tables of two variables. *D. H. Lehmer* (Berkeley, Calif.).

Theory of Sets, Theory of Functions of Real Variables

de Lyra, C. B. A note on Zorn's theorem. Bol. Soc. Mat. São Paulo 4 (1949), 63–65 (1951).

The author attempts to deduce (the well ordering theorem and so) the axiom of choice from Zorn's theorem [cf., e.g., T. Inagaki, Math. J. Okayama Univ. 1, 167–176 (1952); these Rev. 14, 27]. However, the author admits, tacitly, that in each nonvoid set occurring in the proof one can fix a point. *G. Kurepa* (Zagreb).

Słupecki, J. Sur la multiplication des types ordinaux. Colloquium Math. 3, 41–43 (1954).

(1) For any order type $\alpha\neq 0$ there exist order types $\alpha'\neq\alpha$ and $\beta\neq 0$ such that $\beta\cdot\alpha=\beta\cdot\alpha'$. (2) For any order type $\alpha\neq 0$ there exist order types $\alpha'\neq\alpha$ and $\beta\neq 0$ such that $\alpha\cdot\beta=\alpha'\cdot\beta$. In particular: (3) For any order type $\alpha\neq 0$ there exists an order type $\alpha'=\alpha$ such that $\eta\cdot\alpha=\eta\cdot\alpha'$ (where η is the order type of the set of rational numbers in their natural order). (4) For any order type $\alpha\neq 0$ there exists an order type $\alpha'\neq\alpha$ such that $\alpha\cdot(\omega^*+\omega)=\alpha'\cdot(\omega^*+\omega)$. A consequence of (3) and (4) is: (5) For any order type $\alpha\neq 0$ there exist order types α' and α'' such that $\alpha'\neq\alpha$, $\alpha''\neq\alpha$, $(\omega^*+\omega)\cdot\eta\cdot\alpha=(\omega^*+\omega)\cdot\eta\cdot\alpha'$, and $\alpha\cdot(\omega^*+\omega)\cdot\eta=\alpha''\cdot(\omega^*+\omega)\cdot\eta$. *F. Bagemihl.*

Ginsburg, Seymour. On the λ-dimension and the A-dimension of partially ordered sets. Amer. J. Math. **76**, 590–598 (1954).

The paper is connected with one of Komm's [same J. **70**, 507–520 (1948); these Rev. **10**, 22] and one of the author's [Trans. Amer. Math. Soc. **74**, 514–535 (1953); these Rev. **14**, 853]. The λ-dimension of $P'(E_n)$, resp. $P(E_n)$, is $\aleph_0$, resp. c $(2 \leqq n \leqq \omega)$ [for $n < \omega$, see Komm, loc. cit., Cor. 5, 21, resp. Th. 5.3]. A correction of Komm's Th. 6.1 is given. An example of an ordered set P is given whose λ-dimension is 2, while the A-dimension of P equals infinity. Terminology: A linear set E has the A-property, if no disjoint subsets of E, of power c each, are similar. The A-dimension of an ordered set S is the minimum number of total order extensions of S, each of which is similar to a linear set possessing the A-property, the partial order of S being represented by means of these total extensions of S. The set E_n is the set of all the n-sequences of real numbers; $P(E_n)$, $P'(E_n)$ are respectively the orderings of E_n such that for its elements the relation $\{x_i\}_{i<n} < \{y_i\}_{i<n}$ means that $x_i < y_i$ for all the $i < n$, and for at least one $i < n$, respectively.

G. Kurepa (Zagreb).

Novák, Josef. On some characteristics of an ordered continuum. Čehoslovack. Mat. Ž. **2(77)** (1952), 369–386 (1953). (Russian. English summary)

Let C be a continuous ordered set having two end points. Call $\aleph_\alpha$ a lower or an upper character of C according as there exists an element $x \in C$ whose character [cf. Hausdorff, Grundzüge der Mengenlehre, Veit, Leipzig, 1914, pp. 143–144] is $c_{\rho\sigma}$, and $\aleph_\alpha$ equals min $(\aleph_\rho, \aleph_\sigma)$ or max $(\aleph_\rho, \aleph_\sigma)$. Let $P(C)$, $Q(C)$ denote, respectively, the set of all lower, upper, characters of C. Further, let $\mathfrak{a}$ belong to any of the following sets of cardinals if, and only if, $\mathfrak{a}$ is the power of a set of the kind described in parentheses: $\mathfrak{S}(C)$ (monotonic sequence of elements in C), $\mathfrak{J}(C)$ (subset of C that is isolated C), $\mathfrak{J}'(C)$ (disjoint system of nondegenerate intervals of C), $\mathfrak{M}(C)$ (subset of C that is dense in C). Finally, let $\mathfrak{R}(C)$ be the set of all cardinal numbers of orders of dyadic partitions of C [cf. Novák, Fund. Math. **39**, 53–64 (1953); these Rev. **15**, 17]. Define the cardinals $\mathfrak{p}(C)$, $\mathfrak{q}(C)$, $\mathfrak{d}(C)$, $\mathfrak{i}(C)$, $\mathfrak{i}'(C)$, $\mathfrak{r}_2(C)$ to be the respective suprema of the sets $\mathfrak{P}(C)$, $\mathfrak{Q}(C)$, $\mathfrak{S}(C)$, $\mathfrak{J}(C)$, $\mathfrak{J}'(C)$, $\mathfrak{R}(C)$; $\mathfrak{m}(C)$ and $\mathfrak{r}_1(C)$, the respective minima of $\mathfrak{M}(C)$ and $\mathfrak{R}(C)$; and $\mathfrak{n}(C)$, the power of C; the author calls these cardinals "characteristics of C". Then (omitting "C" for brevity)

$$\mathfrak{P} \cup \mathfrak{Q} \subset \mathfrak{S} \subset \mathfrak{J} = \mathfrak{J}', \quad \mathfrak{p} \leqq \mathfrak{q} = \mathfrak{d} \leqq \mathfrak{i} = \mathfrak{i}' \leqq \mathfrak{m} \leqq \mathfrak{n} \leqq \min(\mathfrak{m}^{\mathfrak{d}}, 2^{\mathfrak{r}_1}),$$
$$\mathfrak{d} \leqq \mathfrak{r}_1 \leqq \mathfrak{r}_2 \leqq \mathfrak{m} = \max(\mathfrak{i}, \mathfrak{r}_1) = \max(\mathfrak{i}, \mathfrak{r}_2), \quad \mathfrak{r}_2 \leqq \mathfrak{d}^+,$$

where $\mathfrak{d}^+$ is the least cardinal number greater than every element of $\mathfrak{S}$. The number of elements of C having symmetrical (unsymmetrical) character is $\mathfrak{n}(C)$ $(\leqq \mathfrak{m}(C))$. The set $\mathfrak{R}(C)$ is either $\{\mathfrak{d}\}$, or $\{\mathfrak{d}^+\}$, or $\{\mathfrak{d}, \mathfrak{d}^+\}$, and examples show that each of these cases can occur. If $\aleph_\alpha \leqq \aleph_\sigma$ and $2^{\aleph_0} \leqq \aleph_\sigma$, then there exists a continuous ordered set C such that $\aleph_\alpha = \mathfrak{p}(C) \leqq \aleph_\sigma = \mathfrak{q}(C) = \mathfrak{n}(C)$.

F. Bagemihl.

Novotný, Miroslav. Sur une caractéristique du continu ordonné. Čehoslovack. Mat. Ž. **3(78)**, 75–82 (1953). (Russian. French summary)

(Cf. the preceding review.) The characteristic $\mathfrak{q}(C)$ satisfies the relation $\mathfrak{n}(C) \leqq 2^{\mathfrak{q}(C)}$; it is the smallest power $\mathfrak{a}$ such that every $H \subset C$ of power $> \mathfrak{a}$ contains at least one element dividing H into two subsets, each of power $> \mathfrak{a}$; it is also the smallest power $\mathfrak{b}$ such that every $H \subset C$ of power $> \mathfrak{b}$ contains a subset whose order type is dense. If $\mathfrak{a}$ and $\mathfrak{b}$ are arbitrarily given transfinite powers such that $\mathfrak{a} \leqq \mathfrak{b} \leqq 2^{\mathfrak{a}}$, then there exists a continuous ordered set K such that $\mathfrak{q}(K) = \mathfrak{a}$ and $\mathfrak{m}(K) = \mathfrak{b}$. If C is a continuous ordered set, and $\mathfrak{a}$ is an arbitrary power $< \mathfrak{p}(C)$, then there exists a quasihomogeneous [cf. Novák, Czechoslovak Math. J. **1(76)**, 63–79 (1951); these Rev. **14**, 146], symmetric, continuous ordered set K such that $\mathfrak{q}(K) = \mathfrak{q}(C)$, $\mathfrak{m}(K) = \mathfrak{m}(C)$, and $\mathfrak{a} < \mathfrak{p}(K) \leqq \mathfrak{p}(C)$.

F. Bagemihl (Princeton, N. J.).

Kurepa, George. Real and ordinal numbers as sets of rational numbers. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **8**, 270–279 (1953). (Serbo-Croatian summary)

Both the reals and the ordinal numbers $< \omega_1$ are defined in terms of equivalence classes of subsets of the rational numbers, roughly, as follows. A real number r is the family of those infinite subsets of the rationals which have just one limit point, namely r, in the reals; and an ordinal number $\alpha < \omega_1$ is the family of well-ordered subsets of the rationals which have order type α. Two errors were noted: on p. 271, line 2, read "cardinal" for "ordinal"; on p. 272, line 32, read "to the left of" for "left from".

S. Ginsburg.

Kurepa, Georges. Une généralisation des matrices. C. R. Acad. Sci. Paris **239**, 19–20 (1954).

Given an ordered pair A, B of mathematical systems (sets, spaces, etc.) a function f on $A \times B$ is called a matrix of order (A, B). For a linear U defined on an appropriate class of matrices of order (C, C') a product fg of an f of order (A, C) by a g of order (C', B) can be defined by $fg(a, b) = U(f(a, c)g(c', b))$; for example, when $C = C'$, $U\varphi$ could be $\sum_{c \in C} \varphi(c, c)$ or an integral with respect to some measure on C. This may be the first of a series of papers; beyond the definitions, the only result here is an application of the usual representations of permutations by matrices of 0's and 1's to show that every group G is representable as a multiplicative group of matrices of order (G, G) such that each row and column of each representing matrix has in it precisely one 1.

M. M. Day (Urbana, Ill.).

Besicovitch, A. S. On approximation in measure to Borel sets by F_σ-sets. J. London Math. Soc. **29**, 382–383 (1954).

It is proved that there exists a linear Borel set E such that, for any F_σ-set $A \subset E$, the residual set $E - A$ is of Hausdorff dimension 1.

H. D. Ursell (Leeds).

Eggleston, H. G. A measureless one-dimensional set. Proc. Cambridge Philos. Soc. **50**, 391–393 (1954).

The set considered is the Cartesian product F of a set E with itself, i.e., the set of (x, y) for which x and y are in E, where E is a subset of the real line such that the intersection of E with each non-dense perfect set is at most countable and that the set of differences of pairs of points is the whole real line. Such a set F was studied by Sierpiński [Fund. Math. **24**, 48–50 (1935)]. The author shows that it is an example of a set of Hausdorff dimension 1 such that the relations $0 < \Gamma(F) < \infty$ are satisfied by no outer measure Γ which vanishes for each set consisting of a single point.

L. C. Young (Madison, Wis.).

Srinivasan, T. P. On measurable sets. J. Indian Math. Soc. (N.S.) **18**, 1–8 (1954).

The author gives a brief description of the theory of measurable sets along the lines first indicated by Segal and

shows that the class of sets so described coincides with the sets measurable in the sense of Carathéodory.

P. R. Halmos (Chicago, Ill.).

Porcelli, Pasquale. Concerning integrals. Proc. Amer. Math. Soc. **5**, 395–400 (1954).

The aim of this note is to provide a basis for the method of integral extension due to Riesz [Acta Math. **42**, 191–205 (1919)] for the Stieltjes mean integral proposed by H. L. Smith [Trans. Amer. Math. Soc. **27**, 491–515 (1925), pp. 491–492], viz.

$$M\int_a^b f\,dg=\lim_{\sigma}\sum\tfrac{1}{2}[f(x_{i+1})+f(x_i)][g(x_{i+1})-g(x_i)],$$

where the limit is taken in the sense of successive subdivisions $\sigma\equiv[a=x_0<x_1<\cdots<x_n=b]$. The basic theorem is that if S is a subset of $[a, b]$ of g-measure zero, $f_n(x)$ a sequence of step functions uniformly bounded and converging to zero on $[a, b]-S$, $f_n(x+0)$ and $f_n(x-0)$ converging to zero at points corresponding to points of discontinuity of the monotone nondecreasing function $g(x)$, then $\lim_n M\int_a^b f_n\,dg=0$. In the proof it is tacitly assumed (bottom of p. 399) that for a sequence of step functions satisfying the conditions of the theorem the sequence $V_a^b f_n$ of total variations is bounded in n, which need not be true. The theorem is however valid, even when f_n are bounded functions having at most discontinuities of the first kind.

T. H. Hildebrandt (Ann Arbor, Mich.).

Mergelyan, S. N. On a general notion of convergence of a sequence of functions. Akad. Nauk Armyan. SSR. Soobšč. Inst. Mat. Meh. **5**, 3–12 (1950). (Russian. Armenian summary)

Let k be a property of subsets of $[0, 1]$ and let $\{f_i\}$ be a sequence of measurable functions on the same interval. A function f is called a k-limit of the sequence $\{f_i\}$ if and only if for each set M with property k there is a subsequence $\{f_{i_j}\}$ of $\{f_i\}$ such that $\lim_j f_{i_j}(x)=f(x)$ for each x in M. The author gives an example of k-limit when k is the property k_2 of having precisely 2 points, but devotes most of the paper to the case where k is countability. The principal example is a sequence of analytic functions whose set of k-limits includes all real and complex functions.

M. M. Day (Urbana, Ill.).

Huber, Alfred. Note on a theorem of Ostrowski. Proc. Amer. Math. Soc. **5**, 335–336 (1954).

This is one of a series of recent papers devoted to extensions of a classical theorem of Osgood [e.g., Ostrowski, same Proc. **1**, 648–649 (1950); these Rev. **13**, 121; Pettis, Amer. J. Math. **73**, 602–614 (1951); these Rev. **13**, 217; Gordon and McArthur, ibid. **74**, 764–768 (1952); these Rev. **14**, 362]. The author proves a theorem which is equivalent to the following statement: Let T be a set of real numbers, unbounded above, and for each $t\in T$, let f_t be a real continuous function in (a, b); if $\lim_{t\to\infty} f_t(x)=f(x)$ for every $x\in(a, b)$ and f is continuous, then (a, b) contains a point of uniform convergence (a uniform Cauchy point, in the terminology of Pettis) of $\{f_t\}$. This theorem (as well as that of Ostrowski) is contained in stronger results obtained by Gordon and McArthur in more general situations; in particular, the set of uniform Cauchy points of $\{f_t\}$ is residual in (a, b), and the continuity of f may be omitted from the hypotheses.

W. Rudin (Rochester, N. Y.).

Oliver, H. William. The exact Peano derivative. Trans. Amer. Math. Soc. **76**, 444–456 (1954).

The nth Peano derivative of a function $f(x)$ of one real variable is defined by

$$f(x+h)=f(x)+hf_1(x)+\cdots+h^n[f_n(x)+\epsilon(x, h)]/n!,$$

where $\epsilon(x, h)\to 0$ as $h\to 0$. The author establishes properties of an 'exact' Peano derivative, i.e., of an $f_n(x)$ which exists at all points of an interval. He writes $f\in\mathfrak{M}_n^k$ in (a, b) if for every x and $x+h$ in the interval there exists an x' between x and $x+h$ such that

$$f_k(x+h)-f_k(x)-hf_{k+1}(x)-\cdots-h^{n-k-1}f_{n-1}(x)/(n-k-1)!=h^{n-k}f_n(x')/(n-k)!,$$

and writes $g\in\mathfrak{D}$ if $g(x)$ has the Darboux property (of assuming all intermediate values). He establishes a lemma: If $f_n(x)$ exists and >0 for all $x\in[a, b]$, if $f\in\mathfrak{M}^0_{n-1}$, and $f_{n-1}\in\mathfrak{D}$ then f_{n-1} is strictly increasing and continuous in $[a, b]$. He also deduces a number of results including the following: If f_n exists at every point of $[a, b]$ then $f_n\in\mathfrak{D}$ and $f\in\mathfrak{M}_n^k$ $(0\leq k<n)$ in $[a, b]$. If also $f_n(x)$ is bounded above (or below) then $f_n(x)$ is the ordinary nth derivative of $f(x)$. It is also shown that, when $f_n(x)$ is not bounded above or below, the set of points at which $f_n(x)$ is not the ordinary nth derivative can be any non-dense closed set.

U. S. Haslam-Jones (Oxford).

Zawrosky, A. Generalized concept of differential. Repub. Venezuela. Bol. Acad. Ci. Fís. Mat. Nat. **16**, no. 49, 45–55 (1953). (Spanish)

Germay, R. H. Sur l'application de la méthode des approximations successives à la détermination des fonctions inverses. Bull. Soc. Roy. Sci. Liège **23**, 106–114 (1954).

The author discusses the possibility of using successive approximations to solve the system

$$F_j(x_1, \cdots, x_n; y_1, \cdots, y_n)=0,$$

$j=1, \cdots, n$, for the y's in terms of the x's, in the neighborhood of a point where $F_j=0$. The F_j are assumed to have continuous derivatives, and the Jacobian does not vanish, at the point under consideration. If the F_j are uniform limits of sequences of functions, the solution functions may be obtained in terms of the converging functions. Formal statements and proofs are not given.

R. G. Barile.

Pucci, Carlo. Compattezza di successioni di funzioni e derivabilità delle funzioni limiti. Ann. Mat. Pura Appl. (4) **36**, 1–25 (1954).

The work of this paper is closely related to and in some respects subsumes that of two earlier papers by the author [same Ann. (4) **31**, 129–141 (1950); Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **14**, 471–476 (1953); these Rev. **13**, 730; **15**, 19]. Its principal results deal with a sequence f_n of real-valued functions such that the domain of each f_n is a subset A_n of Euclidean k-space, and such that the sequence A_n converges to a set A. Conditions are given (involving partial increments or partial difference quotients of the functions f_i) under which f_n must have a subsequence biconvergent on A to a function which has partial derivatives of prescribed orders at all points interior to A.

V. L. Klee (Seattle, Wash.).

Vituškin, A. G. Some estimates for variations of sets. Doklady Akad. Nauk SSSR (N.S.) **95**, 433–434 (1954). (Russian)

Vituškin, A. G. On Hilbert's Thirteenth Problem. Doklady Akad. Nauk SSSR (N.S.) **95**, 701–704 (1954). (Russian)

Let E be a closed and G an open subset of Euclidean m-space R^m. Further let $N(x')$ be the number of components in G of the set of $x \in E$ for which a particular selection x' of k coordinates of x is kept fixed. The author defines the kth variation of E in G as the sum of the integrals $\int N(x')dx'$ for all such k-dimensional selections. In two special cases, in both of which G is a cube with edges parallel to the axes, the author gives a simple estimate, from below or from above, for this variation: (1) for some $k \leqq q$ when E has, on $(q+1)$-dimensional coordinate hyperplanes, projections of measure 0 and meets the boundaries of both G and the concentric cube of half the side; (2) for all $k \leqq q$ when E is the image of a mapping of R^q into R^m defined by m polynomials of fixed degree r in each of q variables.

The estimates are used in the second paper to establish the existence of a function F of n variables ($n>2$) with continuous partial derivatives of orders $\leqq l$ ($l \geqq 1$), where F is not expressible as a finite superposition of functions of $n-1$ variables whose $(l-1)$th partial derivatives are Lipschitzian. *L. C. Young* (Madison, Wis.).

Kudryavcev, L. D. On differentiable mappings. Doklady Akad. Nauk SSSR (N.S.) **95**, 921–923 (1954). (Russian)

The paper concerns differentiable mappings of a Euclidean n-dimensional domain into space of the same dimension, and consists of a list of 13 theorems, about half of them variants, or generalizations of theorems of function theory (maximum modulus, Rouché's theorem, etc.) while the rest concern measure and the like. Proofs are omitted and references are inadequate: there is no mention of, or comparison with, similar known results not requiring differentiability. *L. C. Young* (Madison, Wis.).

De Giorgi, Ennio. Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni. Ann. Mat. Pura Appl. (4) **36**, 191–213 (1954).

The paper centres around a version of Green's theorem for sets of points in Euclidean r-space: The author shows this version to be valid if and only if the sets have finite "perimeters." This condition seems somewhat restrictive since, as the author observes, it is not in general satisfied by a set whose boundary is a simple closed hypersurface of finite $(r-1)$-dimensional Hausdorff measure. Although the author's definitions and theorems are given in precise terms, he is able to show that his definition of perimeter coincides with one proposed by Caccioppoli [Atti Accad Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 3–11, 137–146 (1952); these Rev. **13**, 830, 925] for $(r-1)$-dimensional boundary measure. This makes it possible to judge more clearly the precise scope of Caccioppoli's definitions. *L. C. Young.*

Bononcini, Vittorio E. Un teorema di continuità per integrali su superficie chiuse. Rivista Mat. Univ. Parma **4**, 299–311 (1953).

The paper is devoted to establishing the following theorem: In Euclidean 3-space, let S_n be (in the author's terminology) oriented closed surfaces converging to an oriented closed surface S, and suppose the Lebesgue areas uniformly bounded; then for any continuous function $f(p, J)$ linear in J, where p and J vary in 3-space, the surface integral on S_n, interpreted as a Weierstrass integral (with p for the point and J for the Jacobian), tends to the corresponding surface integral on S. The author considers specially the case in which the surface integral is a signed volume and he establishes in this case the isoperimetric inequality.

It should be pointed out that by a closed surface, the author means exclusively a Fréchet surface of the type of the disc with a degenerate boundary (consisting of a single point). The reviewer takes strong exception to this pessimistic terminology by Italian analysts. Does it imply that no further types of closed surfaces can come within the scope of their methods? Or is it suggested that the inclusion of further types would occasion modifications too insignificant and trivial to be worthwhile? *L. C. Young.*

Fullerton, R. E. On the subdivision of surfaces into pieces with rectifiable boundaries. Rivista Mat. Univ. Parma **4**, 289–298 (1953).

Let S be an open non-degenerate Fréchet surface of the type of the 2-cell in Euclidean 3-space having finite Lebesgue area. Using geometrical and topological ideas developed by Cesari and the author, the latter shows the existence of a representation T for S from the unit square Q in the plane and a nested sequence of partitions P_k of Q into closed rectangles R_{ik} with sides parallel to the axes and with disjoint interiors having the following properties. (a) The ratio of the larger to the smaller side of each R_{ik} does not exceed 2. (b) For each positive number ϵ there exists a positive integer n such that the diameter of each R_{ik} is less than ϵ if k exceeds n. (c) For each R_{ik} the image under T of that part of the boundary of R_{ik} which is in the interior of Q is a rectifiable continuous curve. *P. V. Reichelderfer.*

Hosemann, R., und Bagchi, S. N. Begründung einer Algebra physikalisch beobachtbarer Funktionen mittels Faltungsoperationen. II. Fourier-Transformation von Potenzreihen und Multipolreihen. Zusammenhang mit Laplace- und Mellin-Transformation. Eine Erweiterung der Funktionentheorie. Z. Physik **137**, 1–30 (1954).

In continuation of a previous article [Z. Physik **135**, 50–84 (1953); these Rev. **14**, 1072] devoted to a discussion of functions of a real variable as defined only within a non-vanishing "experimental error" and over a finite "domain of observation", the authors discuss in detail the Fourier transformation of the most common singular functions (step functions, Dirichlet functions, etc.). *L. Van Hove.*

Theory of Functions of Complex Variables

McLaughlin, J. E., and Titus, C. J. A characterization of analytic functions. Proc. Amer. Math. Soc. **5**, 348–351 (1954).

Using two lemmas on the vector space of 2×2 matrices over the reals the authors prove the following: Let $\mathfrak{F}$ be a family of continuously differentiable mappings of a plane domain into the plane such that (a) if $f_1, f_2 \in \mathfrak{F}$, then $\lambda_1 f_1 + \lambda_2 f_2 \in \mathfrak{F}$ for all real λ_1 and λ_2; (b) for each $f \in \mathfrak{F}$ the Jacobian of f is non-negative and vanishes only if it is of rank zero; (c) $\mathfrak{F}$ contains a pair of analytic functions f_1 and f_2 such that $\mathrm{Im}\, f_1' f_2' \neq 0$ on D. Then $\mathfrak{F}$ is a linear vector space of analytic functions. *H. L. Royden.*

Schaeffer, A. C. Power series and Peano curves. Duke Math. J. **21**, 383–389 (1954).

Salem and Zygmund [same J. **12**, 569–578 (1945); these Rev. **7**, 378] have shown that some Taylor series which converge absolutely on the unit circle C map C into a curve which covers a square. Piranian, Titus and Young [Michigan Math. J. **1**, 69–72 (1952); these Rev. **14**, 262] have constructed a function $f(z)$ which is regular in $|z|<1$ and continuous in $|z|\leq 1$ and such that (*) each value taken by $f(z)$ in $|z|<1$ is taken by $f(e^{it})$, for some t. The author now shows that if $f(z)=\sum a_n z^{\lambda_n}$, with $a_1\neq 0$, $\sum|a_n|<\infty$, $|a_{n+1}/a_n|\geq\rho>3/4$, and $\lambda_{n+1}/\lambda_n\geq\mu_n$, where μ_n is increasing and $\sum\mu_n^{-1/2}\leq 1/10$, then $f(z)$ has the property (*).

G. Piranian (Ann Arbor, Mich.).

Erdös, Paul, Herzog, Fritz, and Piranian, George. On Taylor series of functions regular in Gaier regions. Arch. Math. **5**, 39–52 (1954).

Die Verff. beschäftigen sich mit Funktionen $f(z)=\sum a_n z^n$, die in einem Kreis C_a: $|z+a|<1+a$ $(a>0)$ regulär sind; C_a wird als 'Gaier disc' (G.d.) bezeichnet. Der Ref. bewies [Math. Z. **56**, 326–334 (1952), insbes. S. 327–328; diese Rev. **14**, 369]: (1) Ist $f(z)$ in C_a regulär und beschränkt, so gilt $a_n=O(n^{-1/2})$; (2) existiert ferner $\lim f(z)$ für $z\to 1$ in C_a, so gilt sogar $a_n=o(n^{-1/2})$. Daran anknüpfend beweisen die Verff. auf kunstvolle Art, dass O in (1) nicht allgemein durch o ersetzt werden kann, und behandeln dann den allgemeineren Fall, dass (3) $(1-z)^k f(z)$ für reelles k in einem G.d. beschränkt ist. Es ergibt sich hier

$$(4)\quad a_n=O(n^{k-1})\quad (k>1);\qquad a_n=O(\log n)\quad (k=1);\qquad a_n=O(n^{(k-1)/2})\quad (k<1);$$

darin kann O nicht allgemein durch o ersetzt werden, jedoch sicher dann, wenn $k\leq 1$ ist und $\lim(1-z)^k f(z)$ für $z\to 1$ in C_a existiert. Sodann werden, wieder unter der Voraussetzung (3), Abschnitte der Form $S_n=\sum_{j=n}^{2n}|a_j|$ wie folgt abgeschätzt:

$$S_n=O(n^k)\quad (k>\tfrac{1}{2});\qquad S_n=O(\sqrt{(n\log n)})\quad (k=\tfrac{1}{2});\qquad S_n=O(n^{k+1})\quad (k<\tfrac{1}{2});$$

für $k\neq\frac{1}{2}$ kann O nicht durch o ersetzt werden. Auf ähnliche Weise werden einige Konvergenzaussagen gewonnen. Gilt (3) für ein $k<-\frac{1}{2}$ [d.h. es ist $f(z)=O((1-z)^\alpha)$ $(\alpha>\frac{1}{2})$ für $z\to 1$ in C_a], so ist $\sum|a_n|<\infty$ und $\sum n|a_n|^2<\infty$; die Aussage wird falsch für $k=-\frac{1}{2}$. Gilt (3) für ein $k<0$, so konvergiert $\sum a_n z^n$ gleichmässig auf $|z|=1$, und auch dies wird falsch für $k=0$. Ist schliesslich $(1-z)^k f(z)$ $(k<1)$ in einem Gebiet regulär und beschränkt, das $|z|<1$ enthält und dessen Rand den $|z|=1$ in $z=1$ von der Ordnung p berührt $(p\geq 1)$, so gilt allgemeiner als in (4) stets $a_n=O(n^{(k-1)/p})$.

D. Gaier (Stuttgart).

Teghem, J. Sur les coefficients tayloriens et les points singuliers des fonctions analytiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 518–525 (1954).

Let $f(z)$ be represented near $z=0$ by the series $\sum_0^\infty a_n z^n$ with positive radius of convergence. The author applies the Laurent transformation $\Pi(\alpha)$ [same Bull. (5) **36**, 730–741 (1950); these Rev. **12**, 695] and the Euler-Knopp transformation $[E,\beta]$, [ibid. **35**, 357–360 (1949); these Rev. **11**, 98] to determine sufficient conditions on the a_n in order that $f(z)$ may have a zero at infinity or a pole at infinity of less than a prescribed order and finally that the origin shall be outside the convex envelope of the set of singularities of $f(z)$. Some simple examples are given.

R. Wilson.

Macintyre, A. J., and Wilson, R. Operational methods and the coefficients of certain power series. Math. Ann. **127**, 243–250 (1954).

Operational methods are used to derive a number of asymptotic expansions from those of the Bessel functions and of the generalized Bessel functions discussed by Wright. Suppose $\psi(z)=\sum c_m z^m=G(1/(1-z))$, where $G(w)$ is an entire function of mean type and finite order ρ. If $f(z)=\sum a_n z^{-n-1}$ is the principal part of the Laurent expansion of $\psi(e^{-z})$ in powers of z, then $F(z)=\sum a_n z^n$ has the property that $c_m=F(m)$ for $m\geq 1$. If we write $\psi(e^{-z})=f_1(z)\cdot g_1(z)$ where $z^{-1}f_1(z^{-1})$ is an entire function and $g_1(z)$ is regular for $|z|<2\pi$ with $g_1(0)=1$, then we find $F(z)=\sum\mu_n F_1^{(n)}(z)$, where $\mu_0=1$ and where $F_1(z)$ is the inverse transform of $f_1(z)$. Under general conditions the asymptotic behavior of $F_1(z)$ determines that of $F(z)$, therefore also that of the coefficients c_m. As an example the authors give the asymptotic behavior of the coefficients c_m in the following cases: (1) $G(z)$ is a polynomial in z; (2) $G(z)=e^{\gamma z}$; (3) $G(z)=\exp P(1/(1-z))$, where P denotes a polynomial (in this case $F_1(z)$ involves Wright's generalized Bessel function);

$$(4)\qquad G(z)=(1-z)^{-\alpha}\exp(\gamma/(1-z))$$

(in this case the Riemann-Liouville integration of fractional order is applied); (5) in the last case the authors combine algebraic, logarithmic and exponential singularities.

J. G. van der Corput (Berkeley, Calif.).

Badalyan, G. V. Generalized factorial series. Akad. Nauk Armyan. SSR. Soobšč. Inst. Mat. Meh. **5**, 13–84 (1950). (Russian. Armenian summary)

The author considers series of the form

$$f(z)=\sum_{n=1}^{\infty}a_n\Big/\prod_{\nu=1}^{n}(1+z/\gamma_\nu),$$

where $\{\gamma_\nu\}$ is an increasing positive sequence and $\sum 1/\gamma_\nu=\infty$. He obtains analogues of many theorems which are familiar for ordinary factorial series: region of convergence, associated Dirichlet series, uniqueness, gap series, conditions for the representability of a given function, etc. The series of R. Lagrange [Acta Math. **64**, 1–80 (1935)] is generalized in a similar way to

$$\sum_{n=1}^{\infty}a_n\prod_{\nu=1}^{n}(1+z/\alpha_\nu)\Big/\prod_{\nu=1}^{n}(1+z/\beta_\nu).$$

One of the author's tools is a generalized Taylor series which may have independent interest. Let

$$\omega_\nu(t)=\int_1^t t_1^{\gamma_1-1}dt_1\int_1^{t_1}t_2^{\gamma_2-\gamma_1-1}dt_2\cdots\int_1^{t_{n-1}}t_n^{\gamma_n-\gamma_{n-1}-1}dt_n;$$

let $\phi(t)$ be given and define $\phi_n(t)$ by

$$\phi_1(t)=\phi'(t),\quad \phi_{n+1}(t)=[\phi_n(t)t^{\gamma_n-1-\gamma_{n+1}}]';$$

then

$$\phi(t)=\phi(1)+\phi_1(1)\omega_1(t)+\cdots+\phi_n(1)\omega_n(t)+R_n(t),$$

$$R_n(t)=\int_1^t t_1^{\gamma_1-1}dt_1\cdots\int_1^{t_{n-1}}t_n^{\gamma_n-\gamma_{n-1}-1}dt_n\int_1^{t_n}\phi_{n+1}(t_{n+1})dt_{n+1}.$$

From this follow a new proof of the completeness of $\{t^{\gamma_n}\}_1^\infty$ when $\sum 1/\gamma_n=\infty$, and a more general theorem.

R. P. Boas, Jr. (Evanston, Ill.).

Walsh, J. L., and Evans, J. P. On approximation by bounded analytic functions. Arch. Math. **5**, 191–196 (1954).

Let R_0 be an open subset of a region R_1 whose boundary C_1 consists of $s(\geq 1)$ mutually disjoint Jordan curves. In previous papers [Walsh, Proc. Nat. Acad. Sci. U. S. A. **24**, 477–486 (1938); Nilson and Walsh, Trans. Amer. Math. Soc. **55**, 53–67 (1944); these Rev. **5**, 115] various results have been proved on approximation to a function $f(z)$ which is analytic on a certain region containing the closure $\bar{R}_0$ of R_0 in its interior, by functions $f_M(z)$ which are analytic and bounded in R_1, as $M\to\infty$; where $|f_M(z)|\leq M$. Here results are extended, to cover the case when C_0, the boundary of R_0, has points in common with C_1. In the proof an essential part is played by a function $\Phi(z)$ which is harmonic and bounded in $\bar{R}_1-R_0-S$, $=0$ on C_0-S and $=1$ on C_1-S; here $S=C_0\cdot C_1$. In the previous results S was empty. From them the existence of $\Phi(z)$ is derived here by an approximation process. Among the corollaries there is a result on approximation to a function which, throughout R_1, is analytic but not bounded. *H. Kober* (Birmingham).

Džrbašyan, M. M. On a new integral transform and its application in the theory of entire functions. Doklady Akad. Nauk SSSR (N.S.) **95**, 1133–1136 (1954). (Russian)

The author states some integral transform formulas involving the function $E_\rho(z;\mu)=\sum_{n=0}^{\infty}z^n/\Gamma(\mu+n/\rho)$, $\rho\geq\frac{1}{2}$. He uses these to generalize the Paley-Wiener theorem on integral representation of entire functions of exponential type which belong to L^2 on a line. He characterizes several classes of entire functions of order ρ and finite type ($\rho\geq\frac{1}{2}$) for which $|f(z)|^2|z|^{2\mu\rho-\rho-1}$ $(\frac{1}{2}<\mu<\frac{1}{2}+\rho^{-1})$ is integrable along two or more rays. *R. P. Boas, Jr.* (Evanston, Ill.).

Lehto, Olli. On the distribution of values of meromorphic functions of bounded characteristic. Acta Math. **91**, 87–112 (1954).

The author considers the distribution of values of a meromorphic function of bounded characteristic in $|z|<1$ under the assumption that, for almost all $e^{i\theta}$ on $|z|=1$, the radial limit values $f(e^{i\theta})$ lie in a given closed set Γ having positive capacity and a non-empty connected complement. The principal result is the following minorization theorem: If $f(0)$ is not in Γ, then for all values α in the complement of Γ, except for a possible set of capacity zero, $N(1,\alpha)=\lim_{r\to 1}N(r,\alpha)\geq g(\alpha,f(0))$, where $g(\alpha,f(0))$ is the Green's function of the compliement of Γ with singularity at $f(0)$, and $N(r,\alpha)$ is the counting function $\int_0^r r^{-1}n(r,\alpha)dr$. It is shown also that either $f(z)$ takes no values outside Γ, or else $f(z)$ takes all values outside Γ except for a possible set of capacity zero. These results are closely related to those published recently by the author [Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **160** (1953); these Rev. **15**, 517]. *A. J. Lohwater* (Ann Arbor, Mich.).

Jenkins, James A. On the local structure of the trajectories of a quadratic differential. Proc. Amer. Math. Soc. **5**, 357–362 (1954).

The author discusses the behaviour of the trajectories $Q(z)dz^2>0$ in the neighbourhood of a point z_0, where $Q(z)$ has meromorphic behaviour. He claims that his method is simpler than that employed by Schaeffer and Spencer [Coefficient regions for schlicht functions, Amer. Math. Soc. Colloq. Publ., v. 35, New York, 1950, chap. 3; these Rev. **12**, 326] for the same purpose. *W. K. Hayman.*

Umezawa, Toshio. The coefficients of meromorphic functions. Duke Math. J. **21**, 355–361 (1954).

Let $f(z)$ be meromorphic in $|z|\leq 1$ and have no zeros and no poles on $|z|=1$. The function $f(z)$ is said to be star-like of order p in the direction of the straight line l if this line is cut in precisely $2p$ distinct points by the image of $|z|=1$ in the w-plane by the mapping $w=f(z)$. Let $D(p)$ be the class of such functions $f(z)$ for which the line l coincides with the diametrical line $f(\zeta)\,0\,f(-\zeta)$, $|\zeta|=1$. Let

$$f(z)=\sum_{-\infty}^{\infty}a_nz^n \quad \text{on} \quad |z|=1,$$

and let

$$f(z)=z^q+\sum_{n=q+1}^{\infty}b_nz^n \text{ about } z=0.$$

Let $f(z)$ have exactly s zeros β_j, $j=1,2,\cdots,s$, and t poles α_i, $i=1,2,\cdots,t$, within the unit circle $|z|<1$ and let $q+s-t=p\geq 1$. Then $|a_n|\leq A_n$ for all n, and $|b_n|\leq B_n$ for $n>q$, where

$$F(z)=\sum_{-\infty}^{\infty}A_nz^n=z^q+\sum_{n=q+1}^{\infty}B_nz^n$$

is a known extremal function. This result generalizes to meromorphic functions the author's earlier result for regular functions [Proc. Amer. Math. Soc. **3**, 813–820 (1952); these Rev. **14**, 260] and lends support to a conjecture that $|b_n|\leq B_n$ for all regular functions p-valent in $|z|<1$ [see A. W. Goodman, ibid. **2**, 349–357 (1951); these Rev. **13**, 22].

For functions star-like of order p in the direction of the real axis the reviewer obtained sharp inequalities of the form

$$(1)\qquad |a_n-\bar{a}_{-n}|\leq\sum_{k=0}^{p}\Delta(p,k,n)|a_k-\bar{a}_{-k}|, \quad n>p,$$

where $\Delta(p,k,n)$ are known polynomials in n depending only upon p and k [Canadian J. Math. **4**, 407–423 (1952); these Rev. **14**, 460]. For the subclass $D(1)$ the author finds

$$(2)\qquad \begin{aligned} |a_{2n+1}-\bar{a}_{-2n-1}|&\leq(2n+1)|a_1-\bar{a}_{-1}|,\\ |a_{2n}-\bar{a}_{-2n}|&\leq 2n|a_1-\bar{a}_{-1}|+|a_0-\bar{a}_0|. \end{aligned}$$

(2) is best possible if $f(z)$ is regular in $|z|\leq 1$ but it is an open question whether (2) is best possible when $f(z)$ has poles in $|z|<1$. Making use of (1), the author extends (2) to the class $D(p)$ but leaves double sums unsimplified, with sharpness an open question again. *M. S. Robertson.*

Umezawa, Toshio. Star-like theorems and convex-like theorems in an annulus. J. Math. Soc. Japan **6**, 68–75 (1954).

The author obtains sufficient conditions for a function $f(z)$, meromorphic and single-valued in $r\leq|z|\leq R$, to be multivalent of order p in this ring. Let $f(z)$ have $n(0)$ zeros and $n(\infty)$ poles in the ring and let

$$\int_{|z|=R}\Re\frac{zf'(z)}{f(z)}d\theta=2p\pi, \quad \int_{|z|=r}\Re\frac{zf'(z)}{f(z)}d\theta=2q\pi.$$

If $\Re[zf'(z)/f(z)]>0$ on $|z|=R$ and $|z|=r$, then $f(z)$ is at most $p+n(\infty)$-valent $(p+n(\infty)=q+n(0))$ and at least $\max[n(0)-0,1]$-valent in $r\leq|z|\leq R$.

Let $F(z)$ be meromorphic and single-valued in $r\leq|z|\leq R$, continuous and $F'(z)\neq 0$ on $|z|=R$ and $|z|=r$. If for a

positive integer k

$$\int_{|z|=r}\left|1+\Re\frac{zF''(z)}{F'(z)}\right|d\theta + \int_{|z|=R}\left|1+\Re\frac{zF''(z)}{F'(z)}\right|d\theta<2\pi(k-n(\infty)+1),$$

then $F(z)$ is convex of order at most $k-n(\infty)-1$ in one direction in $r\leqq|z|\leqq R$. Further, $F(z)$ is at most k-valent and at least $\max[2n(\infty)-k, 1]$-valent in $r\leqq|z|\leqq R$. Further sufficient conditions for multivalency are given involving inequalities for the coefficients of the two Laurent expansions of $f(z)$ (or $F(z)$) holding respectively on the circles $|z|=r$ and $|z|=R$. *M. S. Robertson.*

Kubo, Tadao. Symmetrization and univalent functions in an annulus. J. Math. Soc. Japan 6, 55–67 (1954).

Suppose that $w=f(z)$ maps the annulus $Q<|z|<1$ one-to-one conformally onto a doubly-connected domain bounded by inner and outer continua γ_1, γ_2, respectively. The author shows among other results that if γ_1 is the circle $|w|=Q$, then (i) γ_2 comes nearest to γ_1, when γ_2 is a ray $w>P_Q$ of the positive real axis, (ii) that γ_2 covers the maximum length on the circle $|w|=r$, $P_Q<r<1$, when γ_2 consists of an arc of this circle bisected by the positive real axis together with the ray $w>r$ of this axis, (iii) that if γ_2 consists of the lines $\operatorname{Re} w=0$, 1, then $\operatorname{Im} w$ has the greatest variation on γ_1, when γ_1 is a segment of the line $\operatorname{Re} w=\frac{1}{2}$. The basic tool is symmetrisation of the image-domain and of the harmonic measure of γ_1 or γ_2 in it. Explicit expressions for the extremal functions and constants are given in terms of elliptic functions. In case $Q=0$, (i) reduces to the $\frac{1}{4}$-theorem and (ii) is due to Jenkins [Amer. J. Math. 75, 406–408 (1953); these Rev. 14, 967]. *W. K. Hayman* (Exeter).

Ilieff, Ljubomir. Schlichte Funktionen, die den Einheitskreis konvex abbilden. C. R. Acad. Bulgare Sci. 5 (1952), no. 2–3, 1–4 (1953). (Russian summary)

Let $f_k(z)=z+\cdots$ be regular and univalent in $|z|<1$, and map this region onto a convex region with k-fold symmetry. It is proved that for $k=1, 2, \cdots$ and $|z|\leqq r<1$,

$$|f_k(z)|\geqq r(1+r^k)^{-1/k}, \tag{1}$$

and for $|z_1|\leqq|z_2|\leqq r<1$,

$$|f(z_1)-f(z_2)|\geqq|z_1-z_2|(1+r^k)^{-2/k}. \tag{2}$$

The author also obtains a radius of univalence for the partial sums of the power series. The author asserts that the bounds in (1) and (2) are sharp as shown by the functions $z(1-z^k)^{-1/k}$. The reviewer cannot agree, since for $k\geqq 2$ the image regions of $|z|<1$, under these functions, are not convex. In the proof of (1) the author integrates a strict inequality so the result could not be sharp. The sharp lower bound in (1) was given earlier by M. S. Robertson [Ann. of Math. (2) 37, 374–408 (1936)]. *A. W. Goodman.*

Shah, Tao-Shing. On the distorsion of schlicht functions. Acad. Sinica Science Record 4, 209–212 (1951). (Chinese summary)

Let $F(\zeta)=\zeta+\alpha_1/\zeta+\alpha_2/\zeta^2+\cdots$ be univalent and regular in $|\zeta|>1$, except for the pole at infinity, and let $\sum a_{ij}X_iX_j$ be a real positive definite quadratic form. Goluzin [Mat. Sbornik N.S. 23(65), 353–360 (1948); these Rev. 10, 602] proved that

$$\prod_{i,j=1}^{n}\left|\frac{F(\zeta_i)-F(\zeta_j)}{\zeta_i-\zeta_j}\right|^{a_{ij}}\leqq\prod_{i,j=1}^{n}\left|1-\frac{1}{\zeta_i\bar{\zeta}_j}\right|^{-a_{ij}}.$$

This is now sharpened to

$$\left|\log\prod_{i,j=1}^{n}\left(\frac{F(\zeta_i)-F(\zeta_j)}{\zeta_i-\zeta_j}\right)^{a_{ij}}\right|\leqq\log\prod_{i,j=1}^{n}\left|1-\frac{1}{\zeta_i\bar{\zeta}_j}\right|^{-a_{ij}}.$$

A number of misprints mar the presentation. *A. W. Goodman* (Lexington, Ky.).

Denjoy, Arnaud. Une démonstration de l'identité fondamentale de la fonction $\zeta(s)$ de Riemann. J. Analyse Math. 3, 197–206 (1954).

The author gives a more detailed version of his proof of the functional equation of the Riemann zeta-function [see C. R. Acad. Sci. Paris 238, 533–536 (1954); these Rev. 15, 606]. *E. C. Titchmarsh* (Oxford).

Stallmann, Friedemann. Konforme Abbildung von Kreisbogenpolygonen. I. Math. Z. 60, 187–212 (1954).

The results given are of two kinds. Firstly, one term of each of the asymptotic solutions with respect to ρ of $y''=[r(x)+\rho^2\varphi(x)]y$, in which $x=x_0$ is a regular or irregular singular point, is obtained using the method of Langer. The investigation of the regular singular point case often duplicates work of Langer [Trans. Amer. Math. Soc. 37, 397–416 (1935)] and Cashwell [Pacific J. Math. 1, 337–352 (1951); these Rev. 13, 461]. Secondly, the author uses these results in finding functions mapping a "curvilinear polygon" upon a half plane. In this way a connection is established between the solutions of the equations discussed and the coefficients of the equations. The mapping functions are obtained by the asymptotic integration of the Lamé equation. Equations derived from that of Lamé by confluence are discussed. *N. D. Kazarinoff* (Lafayette, Ind.).

Tsuji, Masatsugu. On Royden's theorem on a covering surface of a closed Riemann surface. J. Math. Soc. Japan 6, 32–36 (1954).

The author is concerned with the type problem of unramified covering surfaces Φ of closed Riemann surfaces F. A test for Φ to be of hyperbolic type is due to Royden [Trans. Amer. Math. Soc. 73, 40–94 (1952); these Rev. 14, 167]. The author gives an alternate proof of this test, and reformulates it as follows. Let C_i, C_i' be disjoint pairs of conjugate 1-cycles on F. Given a subset (C) of the cycles, the open surface $F_0'=F-(C)$ is bordered by edges $\sigma_0^{(i)}$ of the C. One takes an infinite collection of duplicates F_k' of F_0' and joins them successively along edges $\sigma_k^{(i)}$ so as to form an unramified covering surface Φ of F. Then Φ is of hyperbolic type if and only if there exist constant $m_k^{(i)}$ corresponding to the $\sigma_k^{(i)}$ such that: (1) if F_s' and F_k' are adjacent along $\sigma_s^{(i)}$ and $\sigma_k^{(i)}$ respectively, then $m_s^{(i)}=-m_k^{(i)}$; (2) $\sum_i m_0^{(i)}\neq 0$; (3) $\sum_i m_k^{(i)}=0$ $(k=1, 2, \cdots)$; (4) $\sum_{k=0}^{\infty}M_k^2<\infty$, where $M_k=\max_i|m_k^{(i)}|$. *L. Sario* (Los Angeles, Calif.).

Stoll, W. Konstruktion Jacobischer und mehrfachperiodischer Funktionen zu gegebenen Nullstellenflächen. Math. Ann. 126, 31–43 (1953).

Dans l'espace numérique complexe, de dimension p $(u_1, \cdots, u_p)$ muni de sa structure d'espace vectoriel sur le corps des réels, on donne un sous-groupe discret Γ de rang maximum et un diviseur positif $\mathfrak{N}$ invariant par les translations définies par les éléments de Γ. Alors, par la méthode exposée dans Math. Z. 57, 211–237 (1953) [ces Rev. 14, 970], l'auteur construit une fonction de Jacobi k dont le diviseur des zéros est $\mathfrak{N}$. Les vecteurs $\mathfrak{a}$ et $\mathfrak{b}$ de composantes

$(a_1, \cdots, a_p)$, $(b_1, \cdots, b_p)$ étant tels qu'aucune composante de $\mathfrak{R}$ ne soit invariante par toutes les translations λa ou λb (pour $|\lambda| < \lambda_0$), la fonction $f = \sum_{\mu,\nu=1} a_\mu b_\nu \partial^2 \log h / \partial u_\mu \partial u_\nu$ est une fonction méromorphe, périodique dont les périodes sont les éléments de Γ, et si N désigne le support de $\mathfrak{R}$, le diviseur polaire de f est $2N$. L'auteur donne, en particulier, les expressions de h et de f, dans une boule disjointe de N, sous forme d'intégrales sur N. *P. Dolbeault* (Paris).

Behnke, H., und Stein, K. Der Severische Satz über analytische Fortsetzung von Funktionen mehrerer Veränderlichen und der Kontinuitätssatz. Ann. Mat. Pura Appl. (4) **36**, 297–313 (1954)

One of the classical theorems of the theory of analytic functions of several variables is the following. In the space of r real variables $x_1, \cdots, x_r$ and k complex variables $w_1, \cdots, w_k$ $(k \geq 1, r+k \geq 2)$ let G be a bounded domain with connected boundary R. Let $f(x_1, \cdots, x_r, w_1, \cdots, w_k)$ be holomorphic on R. Then f has a holomorphic continuation into the interior of g. The theorem for the case of complex variables only $(r=0)$ is called Hartogs' theorem, and the theorem for the mixed case $(r>0)$, Severi's theorem. Hartogs' theorem with the word "holomorphic" replaced throughout by "meromorphic" is Levi's theorem. All three of these theorems have been investigated by various mathematicians and these three theorems plus the general one, $r \geq 0$, for meromorphic functions, are treated in the paper under review. Most of the proofs of Hartogs', Levi's, and Severi's theorems make use of some form of Cauchy's integral formula, but E. E. Levi and others have related both Hartogs' and Levi's theorems to the continuity theorem. In the present paper the authors state and prove a very general version of the continuity theorem. They also refer to another proof of the continuity theorem for the meromorphic case by W. Rothstein [Arch. Math. **1**, 119–126 (1948); these Rev. **10**, 367]. Having their general form of the continuity theorem, the authors use it to obtain a proof of the Severi theorem for both the holomorphic and meromorphic cases. They also discuss the implications of the theorem for various special types of domains. They then carry over the Hartogs and Levi's theorems to manifolds which the authors here call "complete manifolds". These are the manifolds previously introduced by the second author [Math. Ann. **123**, 201–222 (1951); these Rev. **13**, 224] which are commonly referred to today as Stein manifolds. *W. T. Martin* (Cambridge, Mass.).

Fedorov, V. S Conical monogeneity Mat. Sbornik N.S. **34(76)**, 417–428 (1954). (Russian)

Using the terminology of an earlier paper [Mat. Sbornik N.S. **29(71)**, 177–184 (1951); these Rev. **13**, 226], the author calls a vector function $\vec{f} = (u, v, w)$, where u, v, w are real functions of x_1, x_2, x_3, defined in some region Δ of Euclidean 3-space, conically monogenic in Δ if, for each point M of Δ, the set of directions of monogenicity of $\vec{f}$ consists of the set of all rectilinear generators of some conical surface with vertex at M. Let $u+iv$ be an analytic function of x_1+ix_2 in the square $c < x_k < d$ $(k=1, 2)$, and let w be a differentiable function of x_3 for $c < x_3 < d$. The vector function $\vec{f}$ then has the following properties in the cube $\Delta: c < x_k < d$ $(k=1, 2, 3)$. (1) $\vec{f}$ is completely monogenic at the set N of all points where $u_3 = 0$ and $u_1 = w_3$. (2) If N is the whole cube Δ, then $\vec{f}$ reduces to a linear function of the radius vector $\vec{r} = (x_1, x_2, x_3)$. (3) If N is not the whole cube Δ, then $\vec{f}$ is conically monogenic in $D = \Delta - N$. *A. J. Lohwater.*

Rizza, Giovanni Battista. Contributi al problema della determinazione di una formula integrale per le funzioni monogene nelle algebre dotate di modulo e commutative. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **11**, 134–155 (1952).

Rizza, Giovanni Battista. Estensione della formula integrale di Cauchy alle algebre complesse dotate di modulo e commutative. Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **12**, 667–669 (1952).

Rizza, Giovanni Battista. Teoria delle funzioni nelle algebre complesse dotate di modulo e commutative. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12** (1953), 299–331 (1954).

Dans une étude assez minutieuse l'auteur pose les fondements d'une théorie des fonctions analytiques dans les algèbres commutatives complexes, de dimension finie, munies d élément unité. Soit A une telle algèbre; l'auteur adopte la définition suivante: on dit qu'une fonction $f(x)$, définie dans un domaine D de A et à valeurs dans A, est monogène en un point x_0 de D, s'il existe un $a \in A$ [dit la dérivée de $f(x)$ en x_0] tel que $[f(x)-f(x_0)]/(x-x_0)$ converge vers a quand $x \to x_0$ sur l'ensemble des $x \in D$ pour lesquels $(x-x_0)^{-1}$ existe. En considérant d'abord le cas des algèbres irréductibles et ensuite le cas général, l'auteur retrouve successivement les généralisations du théorème de Cauchy, du théorème fondamental du calcul intégral, de la formule intégrale de Cauchy, de la formule de Taylor, etc. La clef de la théorie est encore, naturellement, la formule intégrale de Cauchy, qui, dans le cas des algèbres irréductibles, est présentée sous la forme

$$\int_\Gamma \frac{f(x)}{x-x_0} dx = 2\pi i N f(x_0),$$

où Γ est un cycle de dimension 1 vérifiant certaines conditions, dont la plus fine concerne l'indice topologique N de laçage de Γ par rapport à l'ensemble des x tels que $(x-x_0)^{-1}$ n'existe pas.

Note du reviewer. Ces résultats se ramènent à la théorie générale des fonctions analytiques dans les algèbres de Banach commutatives complexes, munies d'élément unité [voir E. R. Lorch, Trans. Amer. Math. Soc. **54**, 414–425 (1943); ces Rev. **5**, 100]. Il est visible que l'auteur n'était pas au courant des méthodes abstraites modernes, dont la puissance et la simplicité sont remarquables dans ce domaine. Mais cela n'invalide pas l'intérêt de ses travaux, dont l'originalité essentielle se trouve dans l'étude topologique concernant l'indice N, développée avec l'hypothèse de la dimension finie. *J. Sebastião e Silva* (Lisbonne).

Theory of Series

Macphail, M. S. On some recent developments in the theory of series. Canadian J. Math. **6**, 405–409 (1954).

Verfasser untersucht reversible Matrixverfahren in funktional-analytischer Hinsicht. Seine Ergebnisse sind teilweise schon für normale Matrizen bekannt. Die Resultate im einzelnen lauten: (i) Ein kurzer Beweis des Mazurschen Verträglichkeitssatzes [Studia Math. **2**, 40–50 (1930)] für reversible Verfahren und ein einfaches Verträglichkeitskriterium (beschränkte Spalten bei der Inversen). (ii) Die Eigenschaften PMI und FAK [siehe Wilansky, Duke Math. J. **19**, 647–660 (1952); diese Rev. **14**, 369; Ref., Math. Z. **55**, 55–70 (1951); diese Rev. **13**, 934] sind für reversible

konvergenztreue Verfahren gleichbedeutend. Verfasser stellt noch zwei verwandte Sätze auf. (iii) Ist $(B)\supseteq(A)$ und A permanent und reversible mit der Eigenschaft PMI, so gilt eine Darstellung $B(x)=\rho A(x)+\sum b_k x_k$ für $x \in (A)$.

K. Zeller (Tübingen).

Lorentz, G. G., and Macphail, M. S. Direct theorems on methods of summability. III. Absolute summability functions. Math. Z. **59**, 231–246 (1953).

The authors continue the study of summability functions initiated by the first author [Canadian J. Math. **1**, 305–319 (1949); **3**, 236–256 (1951); these Rev. **11**, 242; **13**, 27]. The definitions and background may be found in the earlier reviews. In the present paper the authors determine the absolute summability functions for the methods of Riesz, Abel and Hausdorff. The new concept of convergence function, defined as follows, is introduced as a tool: let $\sum u_n$ be a positive-term series; $\Omega(n)$ is called a "convergence function" of $\sum u_n$ if $\sum u_{p_i}$ converges whenever $\omega_p(n)\leqq\Omega(n)$. Here $\omega_p(n)$ is the number of terms of the sequence $\{p_i\}$ that do not exceed n.

J. D. Hill (East Lansing, Mich.).

Hill, J. D. Remarks on the Borel property. Pacific J. Math. **4**, 227–242 (1954).

Ein Limitierungsverfahren hat die Borel-Eigenschaft, wenn es (in einem gewissen Sinn) fast alle Folgen aus Nullen und Einsen limitiert. Verfasser gab früher teils notwendige, teils hinreichende Bedingungen für diese Eigenschaft bei Matrixverfahren [Ann. of Math. (2) **46**, 556–562 (1945); Pacific J. Math. **1**, 399–409 (1951); diese Rev. **7**, 153; **13**, 340]. Er verwendet nun diese Bedingungen, um Kriterien für spezielle Verfahrensklassen aufzustellen (Rieszmittel 1. Ordnung, Nörlund- und Hausdorffverfahren). Dabei ergibt sich eine Beziehung zu einem Ergebnis von Lorentz über starke Regularität [Acta Math. **80**, 167–190 (1948); diese Rev. **10**, 367]. Buck und Pollard [Bull. Amer. Math. Soc. **49**, 924–931 (1943); diese Rev. **5**, 117] sowie Tsuchikura [Tôhoku Math. J. (2) **2**, 188–191 (1950); diese Rev. **12**, 820] behandelten Probleme, die die $(C, 1)$-Summierbarkeit von Teilfolgen einer gegebenen Folge betreffen und mit der Borel-Eigenschaft zusammenhängen. Verf. dehnt die Resultate dieser Autoren auf allgemeine Matrixverfahren aus.

K. Zeller (Tübingen).

Korevaar, Jacob. Numerical Tauberian theorems for Dirichlet and Lambert series. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 152–160 (1954).

L'auteur a prouvé [mêmes Proc. **57**, 46–56 (1954); ces Rev. **15**, 698] le résultat suivant: Soit $f(x)=\sum_1^\infty a_n e^{-n^2}$, convergente pour $x>0$, et $\liminf na_n>-\infty$. Si existent s et α positif tels que $f(x)-s=O(1)$ si $x>1$, $=O(x^\alpha)$, si $0<x<1$, alors $\sum_1^n a_p=O(\log n)$, cet ordre de grandeur étant le meilleur. Ce résultat est équivalent au suivant. On suppose $|a_n|=O(n^k)$ $(k>0)$ et $\liminf na_n>-\infty$. Soit $F(z)=\sum_1^\infty a_n n^{-z}$ $(z=u+iv)$. Si pour $u\geqq-\alpha$ $(\alpha>0)$, $F(z)$ est régulière et $|F(z)|=O(\exp\{k'|v|\})$ $(k'>0)$, alors $\sum_1^n a_k-F(0)=O(\log n)$, cet ordre de grandeur étant le meilleur. Résultat analogue en considérant, au lieu de la série de Dirichlet, la série de Lambert $\sum a_n nx/(e^{nx}-1)$.

M. Zamansky (Paris).

Chow, Hung-chi. On the summability $|C|$ of a power series. Bull. Chinese Assoc. Adv. Sci. **1**, no. 5, 30–31 (1953).

A detailed version of this paper appeared in Quart. J. Math. Oxford Ser. (2) **4**, 152–160 (1953); these Rev. **15**, 26.

Pati, T. The summability factors of infinite series. Duke Math. J. **21**, 271–283 (1954).

Die zu einem $f(t)\in L^1(0, 2\pi)$ gehörige Fourierreihe sei mit $\sum(a_n\cos nt+b_n\sin nt)=\sum c_n(t)$ bezeichnet, und $\{\lambda_n\}$ sei eine konvexe Folge positiver Zahlen mit $\sum n^{-1}\lambda_n<\infty$; ferner sei $\Phi(x, t)=\frac{1}{2}[f(x+t)+f(x-t)-2f(x)]$ gesetzt. Chow bewies [J. London Math. Soc. **16**, 215–220 (1941); diese Rev. **4**, 37], dass für fast alle $t\in(0, 2\pi)$ die Reihe (1) $\sum\lambda_n c_n(t)$ $|C, 1|$-summierbar ist. In Satz 1 zeigt nun der Verf., dass unter der lokalen Voraussetzung $\int_0^t\Phi(x_0, t)dt=o(t)$ $(t\to0)$ die Reihe (1) für $t=x_0$ sicher $|C, \alpha|$-summierbar ist $(\alpha>1)$, und verallgemeinert damit ein Resultat von Cheng [Duke Math. J. **15**, 17–27 (1948); diese Rev. **9**, 580], der spezielle λ_n betrachtete. Satz 3: Ist $\sum d_n$ stark $(C, 1)$-summierbar, so ist $\sum\lambda_n d_n$ stets $|C, 1|$-summierbar. Hieraus folgt mit einem Satz von Marcinkiewicz [J. London Math. Soc. **14**, 162–168 (1939); diese Rev. **1**, 11] erneut obiges Resultat von Chow. Schliesslich wird gezeigt: Gehört $g(z)=\sum a_n z^n$ in $|z|<1$ der Klasse L^p an $(1<p\leqq2)$, so ist $\sum a_n(\log n)^{-1-\varepsilon}e^{in\varphi}$ $(\varepsilon>0)$ für fast alle φ $|C, \alpha|$-summierbar $(\alpha>\frac{1}{2})$. Ref. möchte bemerken, dass dies aus einem früheren Resultat von Wang [Duke Math. J. **9**, 567–572 (1942); diese Rev. **4**, 37] folgt.

D. Gaier (Stuttgart).

Sandham, H. F. Some infinite series. Proc. Amer. Math. Soc. **5**, 430–436 (1954).

The author establishes eight identities of which special cases were proved by Ramanujan. Examples:

$$\sum_0^\infty\frac{(-1)^n}{(2n+1)^{4m+1}}\cdot\frac{1}{\cosh(\pi(2n+1)/2)}$$
$$=\frac{\pi}{8}\left(\frac{\pi}{2}\right)^{4m}(E+Ei)^{4m}\quad(m>0),$$

where $1/\cosh x=1+\sum_1^\infty E_n x^n/n!$, and

$$E_n+\binom{n}{1}E_{n-1}E_1 i+\binom{n}{2}E_{n-2}E_2 i^2+\cdots=(E+Ei)^n;$$
$$\sum_1^\infty\frac{n^{4m+1}}{e^{2\pi n}-1}=\frac{1}{2}\frac{B_{4m+2}}{4m+2}\quad(m>0),$$

where $x/(e^x-1)=1+\sum_1^\infty B_n x^n/n!$. The proofs depend on elementary transformations of partial fraction expansions such as

$$\frac{1}{e^{2\pi x}-1}=\frac{1}{2\pi x}-\frac{1}{2}+\frac{1}{\pi}\sum_1^\infty\frac{x}{x^2+n^2}.$$

J. Lehner (Los Alamos, N. M.).

Pisanelli, Domingos. Study of a sequence of powers. Bol. Soc. Mat. São Paulo **6** (1951), 61–68 (1952). (Portuguese)

For all positive values of a, the author evaluates $\lim_{n\to\infty}Q_n$, where $Q_1=a, \cdots, Q_{n+1}=a^{Q_n}, \cdots$.

E. F. Beckenbach.

Aljančić, Slobodan. Asymptotische Entwicklungen A-summierbarer linearer Funktionellen. Srpska Akad. Nauka. Zbornik Radova **35**. Mat. Inst. **3**, 157–212 (1953). (Serbo-Croatian. German summary)

The author proves several theorems on the Abel summability of integrals of the forms

$$\int_0^\infty e^{-iru}d\phi(u)$$

and

$$\int_0^\infty e^{-it\nu}\psi(u)d\phi(u)$$

(where $\psi(t)$ is a Laplace-Stieltjes transform), and on the continuity with respect to the parameter λ of the Abel limits $g(t,\lambda)$ of these integrals when $\phi(u)=\phi(u;\lambda)$. He uses these theorems to show that if $\phi(u;\lambda)$ has an asymptotic expansion (of a generalized kind: not restricted to asymptotic power series) as $\lambda\to\infty$, so (under certain conditions) does $g(t;\lambda)$. In the case where $\phi(u)$ is a step function we have an asymptotic expansion of the Abel sum of a trigonometric series. As examples, the author obtains Stirling's formula; an asymptotic expansion of a Hankel function; and three explicit asymptotic expansions of ultraspherical polynomials which had apparently been worked out previously only in special cases [Szegö, Orthogonal polynomials, Amer. Math. Soc. Colloquium Publ., vol. 23, New York, 1939, theorems 8.21.10, 8.21.11, and formula 8.61.7; these Rev. 1, 14].

R. P. Boas, Jr. (Evanston, Ill.).

Berghuis, J. Some theorems on asymptotic expansions and double integrals. Math. Centrum Amsterdam. Rekenafdeling **MR 11**, 38 pp. (1953).

In this report a number of theorems are collected which are needed in the theory of asymptotic residues which will form the subject of further publications. There are four chapters. Chap. 1 deals with expansion theorems of integrals. A typical example is: Suppose that the integral

$$J(u)=\int_\xi^\eta \varphi(x)dx,$$

where ξ and η are functions of u, exists for small u and that $\xi(u)$ and $\eta(u)$ can be expanded asymptotically for small u. If $\varphi(x)$ can be expanded asymptotically in the neighbourhood of $x=\lim_{u\to 0}\xi$ and $x=\lim_{u\to 0}\eta$, then a constant C can be found such that $J(u)-C\log u$ can be expanded asymptotically for small u. Chap. 2 concerns the asymptotic expansion of the solution $p=p(u)$ of the equation $f(p)=u$, which tends to zero as u tends to zero. Chap. 3 contains some fundamental theorems about double integrals existing in improper sense. Chap. 4 contains various results such as the analytical continuation of the Beta-function.

H. A. Lauwerier (Amsterdam).

Postnikov, A. G. Tauberian theorem for Dirichlet series. Doklady Akad. Nauk SSSR (N.S.) **92**, 487–490 (1953). (Russian)

L'auteur démontre le théorème suivant: supposons que la série de Dirichlet $f(s)=\sum b_n e^{-\lambda_n s}$ ($s=\sigma+it$, $0\leqq\lambda_n\uparrow\infty$) converge absolument pour $\sigma>0$, et supposons que: (1) $|f(s)|$ est borné dans chaque demi-bande $\sigma>0$, $|t|\leqq A$;

$$(2)\qquad \sum_{|\lambda_n-\kappa|>\delta} |b_n|/(\lambda_n-\kappa)^2=O(\log\kappa)$$

et

$$\sum_{|\lambda_n-\kappa|\leqq\delta} |b_n|=O(\log\kappa).$$

Alors $\sum_{\lambda_n\leqq\kappa} b_n=O(\log\kappa)$. *S. Mandelbrojt.*

Tanaka, Chuji. Note on Dirichlet series. X. Remark on S. Mandelbrojt's theorem. Proc. Japan Acad. **29**, 423–426 (1953).

Let $F(s)=\sum a_n\exp(-\lambda_n s)$ have its abscissa of uniform convergence $=-\infty$ and set

$$M(\sigma)=\sup|F(\sigma+it)|,\quad -\infty<t<\infty.$$

The order and type of $F(s)$ is defined as

$$\rho=\limsup_{\sigma\to-\infty}(-\sigma)^{-1}\log^+\log^+ M(\sigma)$$

and

$$k=\limsup e^{\rho\sigma}\log^+ M(\sigma),$$

respectively. Let C be an analytical curve extending to $\Re(s)=-\infty$, let $D(r;C)$ be the curved strip swept by a circle of radius r moving along C. The order and type of $F(s)$ in D is defined by replacing $M(\sigma)$ in the previous formulas by $M(\sigma;D)$, the supremum of $|F(\sigma+it)|$ for s restricted to D. Mandelbrojt [Séries lacunaires, Hermann, Paris, 1936, p. 19] showed that if

$$\liminf(\lambda_{n+1}-\lambda_n)=h>0,\quad \limsup n/\lambda_n=\delta(\leqq 1/h),$$

then in any horizontal strip of width $>2\pi\delta$ the order of the function is the same as in the whole plane. The author now generalizes this result to any curved strip $D(r;C)$ with $r>\pi\delta$ and if $\delta=0$ he shows that $F(s)$ has the same order and type in $D(\epsilon;C)$, $\epsilon>0$, as in the plane. Clever use is made of a lemma which generalizes results of J. J. Gergen and S. Mandelbrojt [ibid., pp. 13–14] according to which $M(\sigma_0)$ is dominated by $A\max|F(u)|$, where A is a constant depending only upon δ, ϵ, and $\{\lambda_n\}$, and u is restricted to a circle $|u-s_1|\leqq\pi(\delta+\epsilon)$ [misprint σ instead of δ in the text] where $\Re(s_1)=\sigma_0-\varphi(\delta,\epsilon,h)$, as well as various estimates which go back to A. Ostrowski. *E. Hille.*

Tanaka, Chuji. Note on Dirichlet series. XI. On the analogy between singularities and order-curves. Proc. Japan Acad. **29**, 478–481 (1953).

For the notation see the preceding review. The author raises the following question: If δ is finite but not necessarily zero, do there exist curves C such that the order of $F(s)$ in $D(\epsilon;C)$, $\epsilon>0$, is the same as in the whole plane? Such a curve is called an order curve. It is now proved that given any curve C extending to $\Re(s)=-\infty$ and denoting by C_r the parallel curve $C+ir$, then there exists a Dirichlet series $\sum b_n\exp(-\lambda_n s)$, obtained by modifying the original series, for which every C_r is an order curve. Here b_n may be chosen either as $\pm a_n$ or such that $|b_n|=|a_n|$ with $\arg b_n-\arg a_n\to 0$ or such that $\arg a_n=\arg b_n$ with $|b_n/a_n|\to 1$. The construction is based on the classical argument first used by A. Hurwitz and G. Pólya for constructing power series with natural boundary and later adapted to Dirichlet series by O. Szász. *E. Hille* (New Haven, Conn.).

Pandey, Nirmala, and Khanna, Girja. On certain Dirichlet's series. Proc. Nat. Acad. Sci. India. Sect. A. **21**, 41–44 (1952).

In this paper the authors prove some theorems regarding conditions under which a general Dirichlet series defines an integral function. The following typical theorem is proved. Let $0<A<2\pi$ and let $f(z)$ and $\lambda(z)$ be analytic functions in some half-plane $\mathrm{Re}(z)\geqq\beta$ (β real), and there satisfy the following restrictions: (a) $\exp(Aif(z))=O(\exp(-k_1|z|))$ for $0\leqq\psi\leqq\pi/2$, $k_1>0$; (b) $\exp(Aif(z))=O(\exp(k_2|z\sin\psi|))$ for $-\pi/2\leqq\psi<0$, $2\pi>k_2>0$; (c) $\lambda(z)=O(|z|)$ as $|z|\to\infty$. Then $\sum_n\exp[Aif(n)-s\lambda(n)]$ defines an integral function of s. *V. F. Cowling* (Lexington, Ky.).

Khanna, Girja. On certain Dirichlet's series. II. Proc. Nat. Acad. Sci. India. Sect. A. **21**, 250–253 (1952).

Let $\phi(z)$ be an analytic function of $z=\beta+\rho e^{i\psi}$ in the half-plane $\mathrm{Re}(z)\geqq\beta$ for some real β. Let

$$\lambda(\psi)=\limsup_\rho[\rho^{-1}\log|\phi(\beta+\rho e^{i\psi})|]\log\rho.$$

Suppose $\lambda(\psi)$ is continuous for $|\psi| \leqq \pi/2$ and $|\lambda(\psi)| \leqq k < \pi$ for $|\psi| \leqq \pi/2$. Then the series $F(s) = \sum_{n}^{\infty} \phi(n) e^{-s\lambda \log n}$, $s = \sigma + it$, $\rho > \beta$, defines an analytic function of s in a domain whose boundary consists of the envelope of the family of lines $\sigma \cos\psi - t \sin\psi = \lambda(\psi)$ where $-\pi/2 \leqq \psi \leqq \pi/2$. For example, if $\lambda(\psi) = A \cos\psi + B \sin\psi$, then the envelope becomes $(\sigma - A)^2 + (t+B)^2 = \epsilon^2$, $\epsilon > 0$, $F(s)$ has the point $A - iB$ as its only finite singularity. *V. F. Cowling* (Lexington, Ky.).

Mikusiński, J. Sur certaines fractions continues finies. Ann. Polon. Math. 1, 203–206 (1954).

Let $K(n)$ denote the maximum number of partial quotients occurring in any one of the simple continued fractions for j/n $(j = 1, 2, \cdots, n-1)$. The author shows by elementary means that $1/2\alpha < K(n)/\log n < 1/\alpha$ $(n = 2, 3, \cdots)$, where $\alpha = \log \frac{1}{2}(1+\sqrt{5})$. It is shown that

$$\limsup K(n)/n = 1/\alpha,$$

and conjectured that $\liminf K(n)/n = 1/2\alpha$.

W. T. Scott (Evanston, Ill.).

Fourier Series and Generalizations, Integral Transforms

Heywood, Philip. A note on a theorem of Hardy on trigonometrical series. J. London Math. Soc. 29, 373–378 (1954).

The author proves that if $\lambda_n \downarrow 0$ and $g(\theta) = \sum_{n=1}^{\infty} \lambda_n \sin n\theta$, if $A > 0$, and $0 < \alpha < 2$, then $\lambda_n \sim A n^{-\alpha}$ if and only if $g(\theta) \sim \{\frac{1}{2}\pi A \theta^{\alpha-1} \csc \frac{1}{2}\alpha\pi\}/\Gamma(\alpha)$ as $\theta \to 0+$. This is a theorem of Hardy [Proc. London Math. Soc. (2) 32, 441–448 (1931)] if $0 < \alpha \leqq 1$; it fails for $\alpha > 2$. *R. P. Boas, Jr.*

Eckart, Gottfried. Über Fourierreihen, die in einem Teilintervall der Entwicklungsperiode identisch verschwinden. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1953, 169–190 (1954).

Identities are obtained on the Fourier coefficients of a periodic function which vanishes in a subinterval. The functions involved are assumed to have a piecewise continuous derivative. *P. Civin* (Eugene, Ore.).

Pati, T. On the absolute Riesz summability of Fourier series and its conjugate series. Trans. Amer. Math. Soc. 76, 351–374 (1954).

Let I^{α} denote integration of order α, with the origin as base. F. T. Wang [Proc. London Math. Soc. (2) 51, 215–231 (1949), pp. 215–217; these Rev. 11, 27] proved that if $I^{\alpha}\{\varphi(t)\} = o\{t^{\alpha}/\log(1/t)\}$ as $t \to +0$, where

$$2\varphi(t) = f(x+t) + f(x-t),$$

then the Fourier series of the periodic and integrable function $f(t)$ is summable by the Riesz means of type $\exp\{(\log \omega)^{1+(1/\alpha)}\}$ and order $\alpha+1$ at $t = x$, where $\alpha > 0$. R. Mohanty [ibid. 52, 295–320 (1951); these Rev. 12, 822] obtained the case $\alpha = 1$ of the following theorem: if $I^{\alpha}\{\varphi(t)\} \cdot t^{-\alpha} \log(k/t)$ is of bounded variation in $(0, \pi)$, where $k > \pi$, then the Fourier series of $f(t)$ is absolutely summable by the Riesz means of the same type and order as above at $t = x$. Here the author gives the case $\alpha =$ a positive integer.

If r is a positive integer, let

$$2g(t) = \{f(x+t) - P(t)\} + (-1)^r \{f(x-t) - P(-t)\},$$

where $P(t)$ is a polynomial of degree $r-1$; for example, if $f(t)$ is r-times differentiable and $P(t)$ is suitably chosen, then $2g(t) = I^r\{f^{(r)}(x+t) + f^{(r)}(x-t)\}$. The following theorem was obtained by Mohanty [loc. cit.] in the case $r = 1$, $\alpha = 1$: If $I^{\alpha-r}\{g(t)\} \cdot t^{-\alpha} \log(k/t)$ is of bounded variation in $(0, \pi)$, then the r-times formally differentiated Fourier series of $f(t)$ is absolutely summable by the Riesz means of the same type and order as above at $t = x$. Here the author gives the case $\alpha =$ an integer $\geqq r$, where r is any positive integer.

Mohanty also obtained results for the conjugate (allied) series and derived conjugate series, which the author extends in a similar way. *L. S. Bosanquet* (London).

Men'šov, D. E. On limits of indeterminacy in measure of partial sums of trigonometric series. Mat. Sbornik N.S. 34(76), 557–574 (1954). (Russian)

Let $f_1(x), \cdots, f_n(x), \cdots$ be a sequence of measurable functions defined almost everywhere in an interval (a, b). We shall say that the measurable function $F(x)$ is the limit superior in measure of $\{f_n(x)\}$ if the following two conditions are satisfied. (1) For any measurable $\phi(x)$ defined almost everywhere in (a, b) we have

$$\lim_{n \to \infty} \text{meas}\{E[f_n(x) > \phi(x)] \cdot E[\phi(x) > F(x)]\} = 0;$$

(2) for any measurable $\psi(x)$ defined almost everywhere in (a, b) and such that meas $E[F(x) > \psi(x)] > 0$ we have

$$\limsup_{n \to \infty} \text{meas}\{E[f_n(x) > \psi(x)] \cdot E[F(x) > \psi(x)]\} > 0.$$

(By $E[f(x) > g(x)]$ we denote here the set of points at which $f(x) > g(x)$.) The limit inferior in measure is defined similarly. The main result of the paper is as follows. Let $G(x)$ and $F(x)$ be measurable on $(-\pi, \pi)$ and satisfy almost everywhere the inequality $G(x) \leqq F(x)$. There exists then a trigonometric series

$$(*) \qquad \tfrac{1}{2}a_0 + \sum_{1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

with coefficients tending to 0 and such that for any sequence $n_1 < n_2 < \cdots < n_k < \cdots$ the partial sums of order n_k of (*) have $G(x)$ for limit inferior in measure and $F(x)$ for limit superior in measure. *A. Zygmund* (Chicago, Ill.).

Timan, M. F. (C, α, β) summability of Fourier series of functions of two variables. Soobščeniya Akad. Nauk Gruzin. SSR 14, 385–392 (1953). (Russian)

Let $f(x, y)$ be an integrable function of period 2π in both x and y, x_0, y_0 a fixed point, $\phi(u, v)$ the mean value of the four quantities $f(x_0 \pm u, y_0 \pm v)$ and, finally,

$$\psi(u, v) = \phi(u, v) - f(x_0, y_0),$$

$$\Psi(u, v) = \int_0^u \int_0^v |\psi(u', v')| \, du' dv'.$$

The author investigates the (C, α, β) summability of the Fourier series of f at the point x_0, y_0 and shows that (1) if $\Phi(u, v) = O(uv)$ as $u, v \to 0$ then the Fourier series of f is, at x_0, y_0, either summable (C, α, β) for each $\alpha > 0$, $\beta > 0$ or for no such α, β. (2) A number of results, too long to be stated here, are obtained about the summability (C, α, β) of the Fourier series of f differentiated termwise, provided f is differentiable (with a suitable definition of differentiability) at x_0, y_0. (3) The following result about numerical series (*) $\sum_0^{\infty} \sum_0^{\infty} u_{mn}$ is used in the proof of (1): suppose that (*) is summable (C, α', β') $(\alpha', \beta' > -1)$ to sum s and suppose that for some $-1 < \alpha < \alpha'$, $1 < \beta < \beta'$, the (C, α, β) means

$\sigma_{m,n}^{\alpha,\beta}$ of (*) are all bounded; then for all γ, $\delta>0$ we have $\sigma_{m,n}^{\alpha+\beta,\beta+\delta}\to s$ $(m, n\to\infty)$. *A. Zygmund* (Chicago, Ill.).

Verblunsky, S. Some theorems on F. A. series. Rend. Circ. Mat. Palermo (2) **3**, 89–105 (1954).

The author establishes further properties of F.A. series which arise in the study of the cooling of a sphere [see S. Verblunsky, Bull. Sci. Math. (2) **76**, 85–96 (1952); these Rev. **14**, 42; L. Fejes, Acta Univ. Szeged. Sect. Sci. Math. **11**, 28–36 (1946); these Rev. **8**, 263]. Let $0=\mu_0<\mu_1<\mu_2<\cdots$ be the roots of the equation $\mu+h\tan\pi\mu=0$, $h>0$, $\mu_{-\nu}=-\mu_\nu$, f a real function in L in $(\alpha, \alpha+2\pi)$ and consider the F.A. series $\sum c_\nu e^{i\mu_\nu x}$ where

$$c_\nu=\lambda_\nu\int_\alpha^{\alpha+2\pi}e^{-i\mu_\nu t}f(t)dt \quad\text{and}\quad \lambda_\nu=\tfrac{1}{2}(h^2+\mu_\nu^2)(h+\pi h^2+\pi\mu_\nu^2)^{-1}.$$

Then $\sum c_\nu e^{i\mu_\nu x}$ is, except for its constant term, an F.A. series in any interval $(\beta, \beta+2\pi)$, and is $(C, 1)$ summable almost everywhere. If $g(x)=\sum_{\nu\neq 0}c_\nu e^{i\mu_\nu x}$, then

$$\lim_{|T|\to\infty}T^{-1}\int_0^T e^{-i\mu x}g(x)dx, \quad \mu\geqq 0,$$

equals c_ν if $\mu=\mu_\nu$ and zero otherwise. If $f\in L^2$, then $\sum|c_\nu|^2\lambda_\nu^{-1}=\int_\alpha^{\alpha+2\pi}f^2(x)dx$. Finally the cases $f\in L^p$ and $\sum|c_\nu|^p<\infty$, $1<p\leqq 2$, are considered and a theorem of Hausdorff-Young type is established. *A. P. Calderón.*

Šneĭder, A. A. On the convergence of Fourier series of Walsh functions. Mat. Sbornik N.S. **34(76)**, 441–472 (1954). (Russian)

The main results of the paper had been stated without proof in an earlier note of the author [Doklady Akad. Nauk SSSR (N.S.) **70**, 969–971 (1950); these Rev. **11**, 353; it was inadvertently omitted in the review that the numbers θ_i figuring there are 0, ± 1]. *A. Zygmund* (Chicago, Ill.).

Avadhani, T. V. On the summability of eigenfunction expansions. I. J. Indian Math. Soc. (N.S.) **18**, 9–18 (1954).

If $f(x_1, \cdots, x_R)$ is L_2 in a bounded domain D, and $\sum a_n w_n(x)$ is an expansion of it in eigenfunctions of the problem $\partial^2u/\partial x_1^2+\cdots+\partial^2u/\partial x_k^2+\lambda u$, $u(x)=0$ on boundary, then

$$\lim_{R\to\infty}\sum_{\lambda_n\leq R}\left(1-\frac{\lambda_n}{R}\right)^\alpha a_n w_n(x^0)=f(x^0)$$

at a point x^0 where the spherical mean of $f(x)$ is continuous at x^0, for $\alpha>(k-1)/2$, and for $\alpha=(k-1)/2$ this is also so if the spherical mean satisfies Lebesgue's criterion there. These two theorems were first stated and proved by this reviewer for multiple Fourier series of periodic functions [Trans. Amer. Math. Soc. **40**, 175–207 (1936)]; the author's proof follows a modification of Titchmarsh, and is briefer than our original one was. However, the reviewer's theorems also applied to functions which are almost periodic in Stepanoff's sense and the fundamental domain D is then no longer a bounded one. *S. Bochner* (Princeton, N. J.).

San Juan, R., and Rodríguez-Salinas, B. Exposition of some known and other new theorems on ordinary and uniform convergence of the Fourier integral. Revista Acad. Ci. Madrid **47**, 495–510 (1953). (Spanish. French summary)

The authors prove some classical results and establish three theorems on the uniform convergence of the partial sums of the Fourier integral $F(x)$ of $f(x)$, assuming that $f(x)\in L^\alpha$, $1<\alpha\leqq 2$, $F(x)$ is absolutely continuous and $F'\in L^{\alpha/(\alpha-1)}$, or that $f(x)\log x\in L^\alpha$ and its Fourier transform is continuous. *A. P. Calderón* (Princeton, N. J.).

Esseen, Carl-Gustav. A note on Fourier-Stieltjes transforms and absolutely continuous functions. Math. Scand. **2**, 153–157 (1954).

Let $f(u)=\int_{-\infty}^{\infty}e^{iuy}dF(y)$ where $F(y)$ is of bounded variation on the whole real line $(-\infty<y<\infty)$. The author proves that if $f(u)\in L_2(-\infty, 0)$ then $F(y)$ is absolutely continuous. *I. I. Hirschman, Jr.* (St. Louis, Mo.).

Rooney, P. G. A generalization of the complex inversion formula for the Laplace transformation. Proc. Amer. Math. Soc. **5**, 385–391 (1954).

The author shows that if $f(s)=\int_0^\infty e^{-st}\Phi(t)dt$, then under the same conditions as those for the complex inversion formula,

$$\Phi(t)=(2\pi i\Gamma(1-\beta))^{-1}\mathrm{P}\int_{\gamma-i\infty}^{\gamma+i\infty}(st)^{-\beta}\,{}_1F_1(1;1-\beta;st)f(s)\,ds,$$

if $\mathrm{Re}\,\beta>0$, $\beta\neq 1, 2, \cdots$. This inversion formula also holds under certain conditions for $\beta=1, 2, \cdots$ and $\mathrm{Re}\,\beta\leqq 0$. The methods of proof are different from those of Erdélyi, who demonstrated a more general equation than the upper inversion formula. *W. Saxer* (Zurich).

Saksena, K. M. Study of a generalised Laplace transform. Proc. Nat. Acad. Sci. India. Sect. A. **21**, 201–208 (1952).

The author considers the integral transformation (due to Varma)

$$(*)\qquad f(s)=\int_0^\infty (st)^{m-\frac{1}{2}}e^{-\frac{1}{2}st}W_{k,m}(st)d\alpha(t),$$

which may be considered as a generalization of the Laplace transform. $f(s)$ is computed for various choices of $\alpha(t)$ and conditions on $\alpha(t)$ are given which insure the convergence of (*). *I. I. Hirschman, Jr.* (St. Louis, Mo.).

Kaushik, S. P. A theorem for the generalised Laplace transform. Proc. Nat. Acad. Sci. India. Sect. A. **21**, 209–212 (1952).

If (*) $F(p)=p\int_0^\infty e^{-px}f(x)dx$, then

$$F(1/p)p^{-\nu+1}=p\int_0^\infty e^{-px}dx\int_0^\infty (x/s)^{\nu/2}J_\nu(2(xs)^{1/2})f(s)ds,$$

a result due to Tricomi. The author generalizes this result by replacing (*) by (**) $F(p)=p\int_0^\infty(2xp)^{-1/4}W_{k,m}(2xp)f(x)dx$. The result is too long to reproduce here.

I. I. Hirschman, Jr. (St. Louis, Mo.).

Poli, L. Quelques images symboliques. Ann. Soc. Sci. Bruxelles. Sér. I. **68**, 13–22 (1954).

This paper contains two corrections of misprints, and a large number of additions, to the tables of Laplace transforms by McLachlan, Humbert, and the author [Mémor. Sci. Math. nos. 100, 2d ed., 113, Gauthier-Villars, Paris, 1950; these Rev. **12**, 408]. The additions consist of a few general "rules", and a number of transform pairs involving the gamma function and related functions, hypergeometric series, and orthogonal polynomials in addition to elementary functions. *A. Erdélyi* (Pasadena, Calif.).

Polynomials, Polynomial Approximations

Koehler, Fulton. Bounds for the moduli of the zeros of a polynomial. Proc. Amer. Math. Soc. **5**, 414–419 (1954).

Let $f(z)=a_0+a_1z+\cdots+a_nz^n$, $a_0\neq 0$, $a_n\neq 0$. Define the Newton diagram of $f(z)$ as the broken line D in the (x, y)-plane with the properties: (1) D extends from the point $x=0$, $y=-\log|a_0|$ to the point $x=n$, $y=-\log|a_n|$; (2) $y_\nu\leqq(y_{\nu-1}+y_{\nu+1})/2$, $\nu=1, 2, \cdots, n-1$, where y_ν is the ordinate of D for $x=\nu$; (3) the points $(\nu, -\log|a_\nu|)$, $\nu=0, 1, \cdots, n$, all lie on or above D; (4) if (ν, y_ν) is a corner point of D, i.e., $y_\nu<(y_{\nu-1}+y_{\nu+1})/2$, then $y_\nu=-\log|a_\nu|$. Values of ν for which $(\nu, -\log|a_\nu|)$ is a corner point or an end point of D are called principal indices of $f(z)$. Let $y_\nu=-\log T_\nu$, so that $T_\nu=|a_\nu|$ when ν is a principal index, and let $R_\nu=T_{\nu-1}/T_\nu$. The slope of D in the interval $(\nu-1, \nu)$ is $\log R_\nu$. Let $z_1, z_2, \cdots, z_n$, be the zeros of $f(z)$ with

$$0<|z_1|\leqq|z_2|\leqq\cdots\leqq|z_n|,$$

and let b_k^n and B_k^n, $k=1, 2, \cdots, n$, be the minimum and maximum respectively of $|z_k|/R_k$ for all polynomials of degree n which do not vanish at the origin. The main result proved here is that b_k^n is equal to the positive root of the equation

$$1=\begin{bmatrix}k\\1\end{bmatrix}x+\begin{bmatrix}k\\2\end{bmatrix}x^2+\cdots+\begin{bmatrix}k\\n-k+1\end{bmatrix}x^{n-k+1}=\phi_k^n(x),$$

where $\begin{bmatrix}k\\p\end{bmatrix}=\binom{k+p-1}{p}$, and ϕ_k^n is a partial sum of the series for $(1-x)^{-k}-1$. This paper is based on one by Ostrowski [Acta Math. **72**, 99–155, 157–257 (1940); these Rev. **1**, 323; **2**, 342]. *E. Frank* (Chicago, Ill.).

Walsh, J. L., and Evans, J. P. Note on the distribution of zeros of extremal polynomials. Proc. Nat. Acad. Sci. U. S. A. **40**, 332–337 (1954).

Let E be a closed bounded point set having a connected complement K and a positive transfinite diameter $T(E)$. Choose a sequence of polynomials

$$p_n(z)=z^n+a_{1n}z^{n-1}+\cdots+a_{nn}$$

of least norm such that, as $n\to\infty$,

$$\lim[\max|p_n(z)|^{1/n}, z \text{ on } E]=T(E).$$

If the zeros of the $p_n(z)$ lie on E or on some closed subset E' containing E, it is known that, as $n\to\infty$,

$$\lim|p(z)|^{1/n}=|f(z)|=\exp[G(z)+\log T(E)],$$

where $w=f(z)$ maps K conformally upon the region $|w|>T(E)$ with $f(\infty)=\infty$ and $G(z)$ is the Green's function (extended sense) for K with pole at ∞. In this paper the authors investigate the case when $K-K'$ (where K' is the complement of E') contains limit points of the zeros of the $p_n(z)$. They find that in this case $\log|f(z)|$ is an exact harmonic majorant of the sequence $[p_n(z)]^{1/n}$; this means that, as $n\to\infty$,

$$\lim[\max|p_n(z)|^{1/n}, z \text{ on } Q]=[\max|f(z)|, z \text{ on } Q]$$

for every bounded continuum Q (not a single point) in $K-\infty$. Furthermore, let C be any analytic Jordan curve lying in K and containing the number c_n of zeros of $p_n(z)$. The authors find that, if C contains only a subset of E on which $G(z)$ has a removable singularity, then, as $n\to\infty$, $\lim(c_n/n)=0$; whereas if C contains a part B of the boundary of K, then as $n\to\infty$,

$$\lim(c_n/n)=(1/2\pi)\int_C(\partial G/\partial\nu)ds=\omega(\infty, B, K),$$

where ν denotes the exterior normal for C and $\omega(\infty, B, K)$ the harmonic measure of B (extended sense) at ∞ with respect to K. The proofs are by application of theory used by Walsh [Duke Math. J. **13**, 195–234 (1946); these Rev. **8**, 201] in the study of sequences of functions approximating to analytic functions. *M. Marden* (Milwaukee, Wis.).

Mikusiński, J. Sur la méthode d'approximation de Newton. Ann. Polon. Math. **1**, 184–194 (1954).

Let c, a, b be positive integers. The approximants x_n to the solution of the equation $x^2-c=0$ are found by Newton's formula

$$(1)\qquad x_{n+1}=\tfrac{1}{2}\left(x_n+\frac{c}{x_n}\right), \quad n=0, 1, 2, \cdots,$$

where x_0 is the largest integer contained in $\sqrt{c}$. The approximants r_n to $\sqrt{c}$ are the approximants of the periodic continued fraction expansion for $\sqrt{c}$. Then $x_n=r_{2^n-1}$ if and only if $c=a^2+2a/b$. In this case the continued fraction is

$$a+\cfrac{1}{b+\cfrac{1}{2a+\cfrac{1}{b+\cfrac{1}{2a+\cdots}}}}.$$

However, formula (1) always gives the approximants of the continued fraction expansion for $\sqrt{c}$ if x_0 is chosen as follows. Let a be the largest integer contained in $\sqrt{c}$ and let the pth partial denominator of the continued fraction expansion for $\sqrt{c}$ be $2a$. The p first partial denominators of this continued fraction constitute a period. This period is primitive when the number $2a$ does not appear before the pth partial denominator. Then, if x_n is taken as the $(p-1)$th approximant of $\sqrt{c}$, $x_{n+1}=r_{2p-1}$. If the primitive period of $\sqrt{c}$ is $2k$, then all the iterations obtained from (1) with x_0 equal to the $(k-1)$th approximant of $\sqrt{c}$ are also approximants of the continued fraction expansion for $\sqrt{c}$.

E. Frank (Chicago, Ill.).

Duncan, W. J. Stability criteria, with special reference to the sextic equation. J. Roy. Aeronaut. Soc. **58**, 431–433 (1954).

This is a discussion of the Routh and Hurwitz stability criteria with particular reference to the characteristic equation of sixth degree. *E. Frank* (Chicago, Ill.).

Gatteschi, Luigi. Il termine complementare nella formula di Hilb-Szegö ed una nuova valutazione asintotica degli zeri dei polinomi ultrasferici. Ann. Mat. Pura Appl. (4) **36**, 143–158 (1954).

In generalizing his previous results on Legendre polynomials [Boll. Un. Mat. Ital. (3) **7**, 272–281 (1952); these Rev. **14**, 466], the author undertakes to obtain "precise" estimates for the Hilb-Szegö formula on ultraspherical polynomials. For the difference

$$\theta^{-1/2}(\sin\theta)^\lambda P_n^{(\lambda)}(\cos\theta)-A(n, \lambda)J_{\lambda-1/2}[(n+\lambda)\theta],$$

$$A(n, \lambda)=2^{\lambda-1/2}(n+\lambda)^{1/2-\lambda}\Gamma(\lambda+1/2)\Gamma(n+2\lambda)[n!\Gamma(2\lambda)]^{-1},$$

the following bounds are obtained:

$$0.696\frac{\lambda(1-\lambda)}{(n+\lambda)^2}\binom{n+2\lambda-1}{n}\theta^{\lambda-1/2} \quad \text{if} \quad 0<\theta\leqq[2(n+\lambda)]^{-1}\pi,$$

$$\frac{\lambda(1-\lambda)}{\Gamma(\lambda)}\{4.075(n/2)^{\lambda-3}\theta^{-1/2}+0.892(n/2)^{\lambda-3}\theta^{1/2}\}$$

if

$$[2(n+\lambda)]^{-1}\pi<\theta\leqq\pi/2.$$

Then $0<\lambda<1$ is assumed. These results are used for the comparison of the zeros of $P_n^{(\lambda)}$ and $J_{\lambda-1/2}$. *G. Szegö.*

Agarwal, R. P. On Bessel polynomials. Canadian J. Math. 6, 410–415 (1954).

Krall and Frink [Trans. Amer. Math. Soc. 65, 100–115 (1949); these Rev. 10, 453] and several other authors have studied properties of the polynomials

$$y_n(x, a, b) = {}_2F_0(-n, a+n-1; -x/b).$$

In the present note it is shown that

$$y_n(x, a, b) = \lim_{\epsilon \to 0} \frac{\Gamma(n+1)\Gamma(\epsilon)}{\Gamma(n+\epsilon)} P_n(\epsilon-1, a-\epsilon-1; 1+2\epsilon x/b),$$

where $P_n(\alpha, \beta; x)$ is Jacobi's polynomial in the customary notation. From this relation the principal properties of y_n follow rather easily using the classical properties of P_n.

G. Szegö (Stanford, Calif.).

Makar, Ragy H., and Makar, Bushra H. Further results on algebraic basic sets of polynomials. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 306–318, 319–330 (1954).

A basic set $\{p_n(z)\}$ is algebraic if its coefficient matrix satisfies an algebraic equation. The authors are interested in connecting the order ω with the number A defined as

$$\lim_{R \to \infty} \limsup_{n \to \infty} \frac{1}{n \log n} \max_{|z|=R} |p_n(z)|.$$

Let $u(n)$ be the degree of p_n. The authors' first result is that if $\{p_n\}$ is algebraic of degree m and $\limsup u(n)/n = u < \infty$ then $\omega \leqq (1+u+\cdots+u^{m-1})A$. Next they estimate the order of the power set $\{p_n\}^k$ for positive or negative integers k, in terms of u, m and A. Under further restrictions on the coefficients the authors estimate the type as well as the order of the entire functions that can be represented by the basic set. Finally the authors introduce quasi-simple monic sets, defined by the property of having a coefficient matrix in which every element in the principal diagonal is 1 while every element below this diagonal is 0, and which has a row-finite reciprocal. Several convergence properties of these sets are discussed.

R. P. Boas, Jr. (Evanston, Ill.).

Mikhail, M. N. Simple basic sets of polynomials. Amer. J. Math. 76, 647–653 (1954).

The author applies the results of a previous paper [Duke Math. J. 20, 459–479 (1953); these Rev. 15, 23] to discuss the order or region of effectiveness of the reciprocal set of a simple set of polynomials or of a product of two simple sets. The general situation is that results previously known with $p_n(z) = z^n + \cdots$ are now proved with $p_n(z) = p_{nn}z^n + \cdots$ provided that $p_{nn} = O(n^k)$ for some fixed k.

R. P. Boas, Jr. (Evanston, Ill.).

Horváth, J. L'approximation polynomiale sur un ensemble non compact. Math. Scand. 2, 83–90 (1954).

Mandelbrojt [Rice Inst. Pamphlet. Special Issue, Houston, Texas, 1951, p. 60; these Rev. 13, 540] has proved the following theorem. Hypotheses: 1) $F(x) > 0$, even, continuous in $-\infty < x < \infty$, $\log F(x)$ convex function of $\log x$; 2) E a closed set on the x-axis, $h > 0$, $E_h = \bigcup_{x \in E}[x-h, x+h]$, $\phi(x)$ characteristic function of the complement of E_h on the x-axis; 3) B, $C > 0$; there is a function $u(x)$ of bounded variation such that $C < u(x) \leqq \frac{1}{2} + \phi(x) + B\phi(x)\phi(-x)$ and

$$\int^{\infty} \log F(e^t) \exp\left[-\int^{t} du/(u(e^t)+u(-e^t))\right] dt = \infty.$$

Then the set $\{x^n/F(x)\}$ $(n = 0, 1, 2, \cdots)$ is fundamental in the space of continuous functions on E tending to 0 at ∞ with the topology of uniform convergence.

The author shows that Mandelbrojt's proof can be considerably simplified by basing it entirely on a lemma of Mandelbrojt's concerning the identical vanishing of functions regular in the complement of E with respect to the z-plane and satisfying an order-condition [loc. cit., p. 34, Lemma M]. The special cases E = entire axis, $E = \{x \geqq \infty\}$ are pointed out, and also that the conditions of Theorem 1 imply completeness in L^p, $p > 0$.

W. H. J. Fuchs.

Vidav, Ivan. Sur une généralisation d'un théorème de M. S. Mandelbrojt. C. R. Acad. Sci. Paris 238, 2138–2140 (1954).

The author modifies the theorem quoted in the preceding review as follows. Hypothesis 1 is changed into: $F(x)$ is continuous; there exists a sequence of polynomials $p_n(x)$ such that for $x \in E$, $|p_n(x)/F(x)| < M$, $\lim_{n \to \infty} p_n(x)/F(x) = 1$. In hypothesis 3 $u(x)$ is supposed to have a derivative which is $o(1)$ at infinity and such that $x^{1/2}u'(x) \in L^2(-\infty, \infty)$. It is assumed that $u(x) = \frac{1}{2}$ on E_h, $u(x) < \frac{1}{2} - \epsilon$ $(\epsilon > 0)$ on CE_h. The integral is replaced by a similar integral extended over the set E only. In many cases this theorem yields a worse result than Mandelbrojt's original theorem (E = {integers}, $E = \{x \geqq 0\}$).

W. H. J. Fuchs (Ithaca, N. Y.).

Orts Aracil, José M.ª. On the region of convergence of a Legendre series. Mem. Real Acad. Ci. Art. Barcelona 31, 329–340 (1953). (Spanish)

A series $\sum_0^\infty a_n P_n(z)$, where the coefficients are in general complex, of Legendre polynomials $P_n(z)$, is found to converge in a certain ellipse. The region of convergence of a series of Tchebycheff polynomials is also found to be an ellipse. Conditions on these series are discussed, as well as consequences of the relation $P_n(z) = \sum \alpha_n^{(k)} T_n(z)$, where the $\alpha_n^{(k)}$ are real and positive such that $\sum \alpha_n^{(k)} = 1$.

E. Frank (Chicago, Ill.).

Special Functions

Delerue, P., et Blondel, J. M. Sur des généralisations de la fonction de Mittag-Leffler. Ann. Soc. Sci. Bruxelles. Sér. I. 68, 42–52 (1954).

The authors discuss several generalizations of Mittag-Leffler's function $E_\alpha(z) = \sum z^n/\Gamma(n\alpha+1)$, the generalizations involving several parameters and variables in various arrangements. They give many properties of these generalized functions.

A. Erdélyi (Pasadena, Calif.).

Meijer, C. S. Expansion theorems for the G-function. VIII. Transformation formulae for generalized hypergeometric functions. Nederl. Akad. Wetensch. Proc. Ser. A. 57 = Indagationes Math. 16, 273–279 (1954).

[For parts I to VII see these Rev. 14, 469, 642, 748, 979; 15, 422, 791.] The general theorems of the earlier part are now shown to contain the known transformations of ${}_2F_1$ and ${}_1F_1$ as particular cases. Two transformations of ${}_pF_q$ are also obtained. These express respectively ${}_pF_q(z)$ and ${}_pF_q(2z)$ as infinite series of series ${}_{p-1}F_{q-1}(z)$ multiplied by powers of z.

A. Erdélyi (Pasadena, Calif.).

Agarwal, R. P. Some relations between basic hypergeometric functions of two variables. Rend. Circ. Mat. Palermo (2) **3**, 76–82 (1954).

This paper contains a number of relations between contiguous basic hypergeometric functions of two variables (recurrence relations), expressions of "basic-partial derivatives" of such functions in terms of contiguous functions, and double contour integral representations.

A. Erdélyi (Pasadena, Calif.).

Lense, J. Über die asymptotische Entwicklung der Hankelschen Funktion für grosse positive Werte der Veränderlichen und des Zeigers. Z. Angew. Math. Mech. **34**, 44–53 (1954). (English, French and Russian summaries)

A discussion of the behavior of $H_\nu^{(1)}(x)$ and $J_\nu(x)$ for large positive real x and ν, $x/\nu=a<1$, is given which is a modification of the saddlepoint approach. The changes stem from alterations in the paths of integration. An oversight of Weyrich [Die Zylinderfunctionen und ihre Anwendungen, Teubner, Leipzig-Berlin, 1937, p. 57] is corrected.

N. D. Kazarinoff (Lafayette, Ind.).

Siegel, K. M., and Sleator, F. B. Inequalities involving cylindrical functions of nearly equal argument and order. Proc. Amer. Math. Soc. **5**, 337–344 (1954).

In an earlier paper [same Proc. **4**, 858–859 (1953); these Rev. **15**, 422] Siegel gave an inequality for $J_\nu(\nu x)$, $0<x<1$ and $\nu>0$. Here similar inequalities are obtained when $1<x<(\nu+2)/(\nu-2)$ by using this result, an inequality for $\partial J_\nu(\nu x)/\partial(\nu x)$, and standard recursion formulas. The range of x may be extended. Bounds of the same type are found for the Neumann function. In each case the bounds remain finite as $x\to 1$. *N. D. Kazarinoff* (Lafayette, Ind.).

Park, David. Asymptotic properties of Bessel functions and the radiation from a synchrotron. J. Math. Physics **33**, 179–184 (1954).

The principal result given is that the function

$$f(\alpha)=J_\nu(\nu+\alpha\nu^{1/3})/J_\nu(\nu)$$

is, asymptotically, a function of α alone. In fact,

$$f(\alpha)=3^{2/3}\Gamma(2/3)\,\mathrm{Ai}(-2^{1/3}\alpha),$$

where $\mathrm{Ai}(x)$ is Airy's function. The derivation hinges upon the expansion of $J_\nu(\nu+\epsilon)$ in powers of ϵ. Application is made to the discussion of the radiation emitted by an electron moving in a synchrotron. *N. D. Kazarinoff.*

Harmonic Functions, Potential Theory

Vidav, Ivan. Sur une extension du théorème de Mandelbrojt-Mac-Lane aux fonctions harmoniques et sous-harmoniques. C. R. Acad. Sci. Paris **238**, 2483–2485 (1954).

A result of Mandelbrojt and MacLane [same C. R. **223**, 186–188 (1946); Trans. Amer. Math. Soc. **61**, 454–467 (1947); these Rev. **8**, 20, 508] shows that a function $F(z)$, which is holomorphic in a band extending to ∞ in one direction, and which decreases to zero with an indicated sufficient degree of rapidity, necessarily satisfies $F(z)\equiv 0$. This result is now extended to harmonic functions and to subharmonic functions. *E. F. Beckenbach.*

Myrberg, Lauri. Über die Integration der Poissonschen Gleichung in einem Kreis. Ann. Acad. Sci. Fennicae. Ser. A. I. Math.-Phys. no. **167**, 20 pp. (1954).

The author considers the Poisson equation $\Delta u=\rho(z)$ in the plane and in the interior of the unit circle. Here ρ is C'. Representations of solutions analogous to Weierstrass infinite products are given. The case where

$$\int^{\infty} M(r)r^{-(\nu+2)}dr<+\infty \quad \left(M(r)=\int_{|z|\leq r}|\rho|\,dxdy\right)$$

is studied in detail as is the corresponding situation in the interior of the unit circle. *M. Heins.*

Hersch, Joseph. "Longueurs extrémales" dans l'espace, résistance électrique et capacité. C. R. Acad. Sci. Paris **238**, 1639–1641 (1954).

Extending the concepts of Ahlfors and Beurling [Acta Math. **83**, 101–129 (1950); these Rev. **12**, 171] to space, the author considers a domain D in space and a distribution function $\rho\geq 0$ therein. For a curve c we define $C_\rho(c)=\int\rho ds$, for a surface γ we define $\Gamma_\rho(\gamma)=\int\int\rho dS$ (lower Darboux integral); finally, we define $V_\rho(D)=\overline{\int\int\int}\rho^2 dV$ (upper Darboux integral). Let $\{c\}$ be a family of curves exhausting the domain D; we define $M\{c\}=\inf_\rho V_\rho$ where ρ is subject to the condition $C_\rho(c)\geq 1$ for each c. Similarly, if $\{\gamma\}$ is a family of surfaces exhausting D, we define $M\{\gamma\}=\inf_\rho V_\rho$ where $\Gamma_\rho(\gamma)\geq 1$ for each γ. The first result of the note refers to a domain D which is of the topological character of the sphere; let G_0 and G_1 be two disjoint domains on the boundary, $\{c\}$ the family of curves within D joining G_0 and G_1, $\{\gamma\}$ the family of surfaces (discs) separating G_0 and G_1 having their boundary curve on the boundary of D. Then $M\{c\}\cdot M\{\gamma\}=1$; in case ρ is constant, $M\{\gamma\}$ is, apart from a trivial factor, the electric resistance of D. In case D is equivalent to a domain between two concentric spheres, the electric capacity of a condenser can be defined in a similar fashion. *G. Szegö* (Stanford, Calif.).

Jenkins, James, and Morse, Marston. Curve families F^* locally the level curves of a pseudoharmonic function. Acta Math. **91**, 1–42 (1954).

Let Σ^* be the sphere with north and south poles removed, and let F^* be a family of curves on Σ^* so that exactly one member of F^* goes through each point on Σ^* and such that the members of F^* are locally the level curves of a pseudoharmonic function. The principal problem then arising is to determine by topological means whether or not there exists a multivalued pseudoharmonic function u on Σ^* with F^* for level curves and satisfying $u(p')=au(p)+b$ when p' and p are two corresponding points on neighboring sheets of the universal covering surface of Σ^*. In the present paper the authors decompose Σ^* into a number of canonical regions determined by F^* and discuss in detail the topological situations which can occur. These results are to be applied in a later paper to give a definitive answer to the above problem.

H. L. Royden (Stanford, Calif.).

Differential Equations

Hartman, Philip, and Wintner, Aurel. On curves defined by binary non-conservative differential systems. Amer. J. Math. **76**, 497–501 (1954).

The author considers the equation

$$(p(t)x')'+h(t)x'+q(t)x=0$$

and shows that if solutions of this equation are non-oscillatory for t large and if as $t\to\infty$, $p(t)\to 1$, $h(t)\to 0$, and $q(t)\to 0$, every nonnull solution of the differential equation has the property that $x'(t)/x(t)\to 0$ as $t\to\infty$. *W. Leighton.*

Sears, D. B. Some properties of a differential equation. II. J. London Math. Soc. **29**, 354–360 (1954).

[For part I see same J. **27**, 180–188 (1952); these Rev. **13**, 744.] Let $f(x)$ be continuous for large x, say $x\geqq X$, and satisfy the inequality (1) $|f(x)|\leqq g(x)+\epsilon(x)$, where $g=g_{n,p}(x)$ is given by

$$\frac{p+1}{x^2p^2}+\frac{p+2}{x^2p^2}\sum_{r=1}^{n}(\log x\log_2 x\cdots\log_r x)^{-1}$$

and $x\epsilon(x)$ is of class $L(X,\infty)$. The author proves that if $p\geqq 1$, $n=0,1,\cdots$, then no solution $\varphi=\varphi(x)\not\equiv 0$ of (2) $\varphi''-f\varphi=0$ is of class $L^p(X,\infty)$ and that, in this assertion, the constants $(p+1)/p^2$, $(p+2)/p^2$ are the best possible. The proof depends on the fact that if $g(x)\geqq 0$ is any continuous function (not necessarily $g_{n,p}(x)$), ψ ($\not\equiv 0$) is a bounded solution of (3) $\psi''-g\psi=0$ and $x\epsilon(x)$ is of class $L(X,\infty)$, then $\varphi=o(\psi)$, as $x\to\infty$, is incompatible with (1) and (2) when $\varphi\not\equiv 0$. It is then shown that if $g=g_{n,p}$ in (1) and (3) and if φ ($\not\equiv 0$) satisfies (2) and is of class $L^p(X,\infty)$, then $\varphi=o(\psi)$ as $x\to\infty$, where $\psi=(x\log x\log_2 x\cdots\log_n x)^{-1/p}$. *P. Hartman* (Baltimore, Md.).

El'šin, M. I. Qualitative investigation of a system of two linear homogeneous equations of the first order. Doklady Akad. Nauk SSSR (N.S.) **94**, 5–8 (1954). (Russian)

The author investigates the possibility of representing the solutions of the second-order linear system, $x'=ax+by$, $y'=cx+dy$, in the form $x=\rho\cos\theta$, $y=\rho\sin\theta$. This may be done in certain cases using the solution of an associated Riccati equation. Some consequences are discussed. *R. Bellman* (Santa Monica, Calif.).

El'šin, M. I. The method of comparison in the qualitative theory of an incomplete differential equation of second order. Mat. Sbornik N.S. **34(76)**, 323–330 (1954). (Russian)

Taking the equation $x''=f(x,x')$ in the form $x'=y$, $y'=f(x,y)$, the author shows that useful results concerning this equation can be obtained by considering the behavior of the solutions of the comparison equations, $x'=u$, $u'=F(x,u)$, $x'=v$, $v'=G(x,v)$, where $F(x,y)/y<f(x,y)/y<G(x,y)/y$ for all x and y. *R. Bellman* (Santa Monica, Calif.).

Lidskiĭ, V. B. On the number of solutions with integrable square of the system of differential equations $-y''+P(t)y=\lambda y$. Doklady Akad. Nauk SSSR (N.S.) **95**, 217–220 (1954). (Russian)

Considered is the system (S): $-y''+P(t)y=\lambda y$, where $y=(y_1,\cdots,y_k)$ is a vector, P is a symmetric matrix of real functions which are summable in each finite subinterval of $0\leqq t<\infty$, and λ is a complex parameter. If $f=(f_1,\cdots,f_k)$, $g=(g_1,\cdots,g_k)$ let $(f,g)=\sum f_jg_j$. Suppose

$$(P(t)h,h)\geqq -q(t)(h,h)$$

for all k-vectors h, where q is a continuous function satisfying $q(t)>\delta>0$ and $\int_0^\infty q^{-\frac{1}{2}}(t)dt=\infty$. Let, in addition, q satisfy either of the two conditions: (1) q is monotone, (2) q is differentiable and $\limsup_{t\to\infty}|q'(t)|/q^{\frac{3}{2}}(t)<\infty$. Then the system (S) has exactly k linearly independent solutions for any non-real λ which satisfy $\int_0^\infty(y,y)dt<\infty$. A second theorem asserts that, if P is differentiable, and for any k-vector h and for all $t>a$,

$$(P(t)h,h)<0,\quad (P'(t)h,h)\leqq(2+\epsilon)t^{-1}(P(t)h,h)\quad(\epsilon>0),$$

then all solutions of (S) satisfy $\int_0^\infty(y,y)dt<\infty$. These results represent extensions of those proved by Sears for the case $k=1$ [Canadian J. Math. **2**, 314–325 (1950); these Rev. **12**, 102], and use of the Sears' results is made in the present proofs. *E. A. Coddington* (Los Angeles, Calif.).

Bautin, N. N. On periodic solutions of a system of differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 128 (1954). (Russian)

The author considers the particular system

$$x'=x(a+bx+cy),\quad y'=y(d+ex+fy),$$

and shows that one can in this case determine when and only when periodic solutions exists. *R. Bellman.*

Kuz'min, P. A. On the theory of stability of motion. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 125–127 (1954). (Russian)

The author considers the linear system, $x'=P(t)x$, where $P(t)$ is a continuous matrix function of t. Using a generalized Liapounoff function, he obtains a number of stability theorems. *R. Bellman* (Santa Monica, Calif.).

Massera, Jose L. Total stability and approximately periodic vibrations. Facultad de Ingeniería Montevideo. Publ. Inst. Mat. Estadística **2**, 135–145 (1954). (Spanish. English summary)

The system (*) $\dot{x}=F(x,t)$ is studied with x an n-component vector, F smooth enough to ensure existence and uniqueness of solutions with given initial data, and $F(0,t)\equiv 0$. The solution $x=0$ is totally stable if, given $\epsilon>0$, there exists a $\delta>0$ such that if $\|F_1-F\|\leqq\delta$ for $\|x\|\leqq\epsilon$ and $t\geqq 0$, and $\dot{x}_1(t)=F_1(x_1(t),t)$ with $\|x_1(0)\|\leqq\delta$, then $\sup_{t\geqq 0}\|x_1(t)\|\leqq\epsilon$. As the author notes, this definition agrees with that of stability under constantly acting perturbations studied by I. Malkin [Akad. Nauk SSSR. Prikl. Mat. Meh. **8**, 241–245 (1944) = Amer. Math. Soc. Translation no. **8** (1950); these Rev. **7**, 298; **11**, 439]. Malkin's sufficient condition for total stability is also established here. Moreover, if F is T-periodic in t and (*) has a solution x that is T'-periodic in t with T' and T incommensurable, then in order that x be totally stable it is sufficient, under certain other conditions, that the form $\sum y_iy_j\partial f_i(x(t),t)/\partial x_j$ be negative definite for $t\geqq 0$. Other theorems involve orbital stability, and approximate periodicity in the following sense: $x(t)$ is ϵ^+-T-periodic if given $\epsilon>0$ there exists a sequence t_n, with $t_0=0$ and $0<T-\epsilon<t_{n+1}-t_n<T+\epsilon$, such that $|x(t_n+\tau)-x(\tau)|<\epsilon$ for $0\leqq\tau\leqq t_{n+1}-t_n$. *F. A. Ficken* (Knoxville, Tenn.).

Barbašin, E. A., and Krasovskiĭ, N. N. On the existence of Lyapunov functions in the case of asymptotic stability in the large. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 345–350 (1954). (Russian)

Consider a system

$$\dot{x}=X(x,t) \tag{1}$$

where x, X are n-vectors, $X(0,t)=0$ for $t\geqq t_0$ and the $\partial X_i/\partial x_j$ are continuous for $t\geqq t_0$ and all x. Theorem 1: In order that (1) be uniformly stable in the large a necessary and sufficient condition is the existence of a function $v(x;t)$ such that: (a) there exists a continuous unbounded function $w(x)$, positive except at the origin where it vanishes, such that $v(x;t)\geqq w(x)$ for $t\geqq t_0$ and all x; (b) there exists a

continuous function $f(x)$ such that $f(0)=0$ and that $v(x;t)<f(x)$ for $t\geqq t_0$ and all x; (c) the total derivative dv/dt is definite negative (zero only at the origin) for $t\geqq t_0$ and all x.

The system (1) is said to be retardation-stable (r.s.) whenever for any $\epsilon>0$ there exist $\delta(\epsilon)>0$, $h>0$, such that given any system of functions $\eta_{ij}(t)$, with $|\eta_{ij}|<h$ for $t\geqq t_0$, then any solution of the system

$$(2)\quad \dot{x}_i=X_i(x_1(t-\eta_{i1}),\cdots,x_n(t-\eta_{in}),t),\quad i=1,2,\cdots,n,$$

starting for $t\geqq t_0$ in the δ-neighborhood of the origin remains in its ϵ-neighborhood for all $t\geqq t_0$.

Let the origin be stable in the large for (1). The system is said to be r.s. in the large whenever given R, $\epsilon>0$, there corresponds an h such that with the η behaving as above any solution of (2) starting within the R-neighborhood of the origin remains within its ϵ-neighborhood. Theorem 2: If the origin is uniformly stable in the large for (1) then the origin is r.s. in the large for (1). *S. Lefschetz.*

Putnam, C. R. Stability and almost periodicity in dynamical systems. Proc. Amer. Math. Soc. **5**, 352–356 (1954).

A solution $x(t)$ of a system of differential equations $x'=f(x)$ (where $x=(x_1,\cdots,x_n)$ and $f=(f_1,\cdots,f_n)$, each f_i being of class C') is {A-stable} [B-stable] with respect to an invariant set Ω containing $x(t)$ provided that corresponding to $\epsilon>0$ there exists $\delta>0$ such that {$|x(0)-y(0)|<\delta$ with $y(0)$ in Ω} [$|x(t_0)-y(t_0)|<\delta$ for any t_0 and with $y(t_0)$ in Ω] implies $|x(t)-y(t)|<\epsilon$ for all t. B-stability (which is essentially uniform equicontinuity) implies A-stability, but a simple example shows that the converse is not true. If Ω is compact and the closure of a B-stable motion $x(t)$, it is known that $x(t)$ is almost periodic in the sense of Bohr. It is shown that if Ω is compact, invariant and possesses a finite invariant measure which is positive for open sets and $x(t)\in\Omega$ is A-stable then: 1) $x(t)$ is recurrent (stable in the sense of Poisson); 2) $\lim_{T\to\infty} T^{-1}\int_0^T g(x(t))dt$ exists for any g continuous on Ω; 3) $x(t)$ possesses an asymptotic distribution function. 1) follows from the Poincaré Recurrence Theorem, which implies that the recurrent points are dense in Ω. (2) follows from the Birkhoff Ergodic Theorem which implies that the points for which the time averages exist are dense in Ω. *G. A. Hedlund* (New Haven, Conn.).

Troickiĭ, V. A. On the behavior of dynamical systems and systems of automatic regulation having several regulating organs near to the boundary of a region of stability. Akad. Nauk SSSR. Prikl. Mat. Meh. **17**, 673–684 (1953). (Russian)

It was shown by N. N. Bautin [Behavior of dynamical systems near the boundary of their region of stability, Gostehizdat, Moscow, 1949] that the question described in the title of his paper was reducible to the investigation of the critical systems of Lyapunov [Problème générale de la stabilité du mouvement, Princeton, 1947; these Rev. **9**, 34], where there is either a single zero characteristic root or else one pair of pure complex such roots. Everything then depends according to Lyapunov upon the parity of the lowest terms of a certain power series and upon the sign of a certain number g. A similar result was obtained for automatic control systems with a single regulating element by A. I. Lur'e [Akad. Nauk SSSR. Prikl. Mat. Meh. **14**, 371–382 (1950); Some nonlinear problems in the theory of automatic controls, Gostehizdat, Moscow-Leningrad, 1951; these Rev. **12**, 181; **15**, 707]. This is now extended to automatic control systems with several regulating organs. Explicit expressions are actually given for the number g. *S. Lefschetz.*

Vinograd, R. È. A new proof of Perron's theorem and certain properties of regulating systems. Uspehi Matem. Nauk (N.S.) **9**, no. 2(60), 129–136 (1954). (Russian)

In connection with investigations of the stability of slightly non-linear systems of the form $x'=A(t)x+f(x,t)$, it is important to have transformations of the form $x=U(t)y$ which reduce $x'=A(t)x$ to $y'=S(t)y$ where $S(t)$ is semi-diagonal or diagonal, and where $U(t)$ and $U'(t)$ possess certain boundedness and inversion properties. A result of this type was first obtained by Perron [Math. Z. **32**, 465–473 (1930)]. The author gives a new proof of Perron's theorem and some further results. *R. Bellman.*

Alzerman, M. A., and Gantmaher, F. R. Conditions for the existence of a region of stability for a single-contour system of automatic regulation. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 103–122 (1954). (Russian)

The authors consider the system of differential equations $d_{1k}(p)x_k=d_{2k}(p)x_{k-1}$, $k=1,2,\cdots,m$; $x_0=-x_m$, $p=d/dt$, where all the operators are of the form ap^2+bp+c. The problem of ascertaining the stability of the differential system leads to the algebraic problem of determining the location of the roots of the associated characteristic equation. This is investigated in great detail. *R. Bellman.*

***Lefschetz, S. On Liénard's differential equation.** Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 149–153. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00.

A discussion of the equations $\dot{x}=y-F(x)$, $\dot{y}=-x$ by means of their extensions in the whole projective plane leads to the result that paths outside the last limit cycle tend spirally to it. The conditions on $F(x)$ are such that there are a finite number of limit cycles about the origin, the outermost being stable outside. For the related equations $\dot{x}=y-F(x)$, $\dot{y}=-x+e(t)$, where $e(t)$ is continuous and has period τ, a method of construction of the necessary simple closed curve representing a solution of period τ is given. This construction also requires knowledge of the behavior of the paths "at infinity", which is obtained by the same extension method used in the homogeneous case.

H. Cohen (Pittsburgh, Pa.).

Stellmacher, K. L. Über erzwungene nicht-lineare Schwingungen hoher Erregerfrequenz und ihre Stabilität. Z. Angew. Math. Mech. **34**, 105–119 (1954). (Russian summary)

The author is concerned with non-resonant vibrations in a system whose behavior may be described by the nonlinear differential equation

$$\frac{d^2\varphi}{d\tau^2}=\sum_{k=1}^{\infty}\sum_{l=-k}^{k}\alpha^k e^{2il\tau}F_{kl}(\varphi)$$

where α is a parameter. The right-hand side is assumed analytic in the three variables. He solves this in the form of a double series in powers of α and $e^{2i\tau}$. The recurrence formulas for the coefficients become, as usual, more complicated with higher powers. The parameters exhibit points of bifurcation between which different numbers of these periodic solutions exist. The convergence of the solution series is established.

The author then investigates the stability of the solutions in the weak sense by the usual linearized perturbation development. The resulting Hill-type equation is not solved in the usual way, but by a power series development similar to that used originally to obtain φ. He is able to do this because he does not attempt to solve for the stability parameter μ (according to the Floquet theory, the solution to the stability equation is of the form $e^{i\mu\tau}v$, where v is periodic in τ) exactly, but expands it in powers of α and obtains recurrence formulas for its coefficients, as well as for the coefficients of v, expanded as a double series in α and $e^{2i\tau}$. The convergence of this process is then established. Finally, the theory is applied to the interesting problem in which the support point of a swinging pendulum is subjected to harmonic oscillations. *E. Pinney.*

Popov, E. P. On approximate investigation of auto-oscillations and forced oscillations of non-linear systems. Doklady Akad. Nauk SSSR (N.S.) **95**, 943–946 (1954). (Russian)

Application of the Krylov-Bogolyubov type of linearization by harmonic balance to a wide variety of nonlinear systems in two variables. *S. Lefschetz.*

Putnam, C. R. On the continuous spectra of singular boundary value problems. Canadian J. Math. **6**, 420–426 (1954).

Let $p(t)>0$, $f(t)$ real-valued, and both p and f continuous on $0\leq t<\infty$. Let $(px')'-fx=0$ be of the limit-point type. Finally, let $\rho_\alpha(\lambda)$ denote the continuous monotone function occurring in Weyl's formula for the spectral resolution of the self-adjoint operator associated with a homogeneous boundary condition $x(0)\cos\alpha+x'(0)\sin\alpha=0$ [Math. Ann. **68**, 220–269 (1910), pp. 238–251]. The author shows that if, for some pair of values $\alpha=\theta$, φ $(\neq\theta)$ on $0\leq\alpha<\pi$, both $\rho_\theta(\lambda)$ and $\rho_\varphi(\lambda)$ are strictly increasing on some λ-interval L and $\rho_\theta(\lambda)$ is absolutely continuous with respect to $\rho_\varphi(\lambda)$ on L, then, for every α, $\rho_\varphi(\lambda)$ is strictly increasing and $\rho_\theta(\lambda)$ is absolutely continuous with respect to $\rho_\alpha(\lambda)$ on L.
P. Hartman (Baltimore, Md.).

Sears, D. B. Integral transforms and eigenfunction theory. Quart. J. Math., Oxford Ser. (2) **5**, 47–58 (1954).

By a variant of the procedures (used by Titchmarsh, Levinson, Yosida and Levitan) of considering a non-singular Sturm-Liouville problem on a bounded interval $0\leq t\leq T$ and then letting $T\to\infty$, the author obtains the Weyl expansion theorems, including the Titchmarsh formulae, for self-adjoint boundary-value problems on $0\leq t<\infty$. The variant involves a theorem on integral transforms which define unitary mappings of a Hilbert space onto another Hilbert space. The author then uses a form of the Parseval relation to prove a theorem of Hartman and Wintner [Amer. J. Math. **70**, 309–316 (1948); these Rev. **10**, 120]; a simple proof, not involving spectral resolutions, for the theorem in question is implied by the result of Hartman [ibid. **71**, 915–920 (1949); these Rev. **11**, 438]. *P. Hartman.*

Naimark, M. A. Investigation of the spectrum and the expansion in eigenfunctions of a nonselfadjoint operator of the second order on a semi-axis. Trudy Moskov. Mat. Obšč. **3**, 181–270 (1954). (Russian)

This is a detailed exposition of results announced in Doklady Akad. Nauk SSSR (N.S.) **85**, 41–44 (1952); **89**, 213–216 (1953); these Rev. **14**, 473; **15**, 33.
E. A. Coddington (Los Angeles, Calif.).

Zwahlen, Robert. Ein "neues" Eigenwertproblem. Actes Soc. Helv. Sci. Nat. **133** (1953), 60–65 (1954).

The author provides examples of differential equations with a parameter whose "eigensolutions" may be obtained consecutively by an extension of an operational device due to Dirac. *W. Leighton* (Pittsburgh, Pa.).

Keller, Herbert B., and Keller, Joseph B. Lowest eigenvalues of nearly circular regions. Quart. Appl. Math. **12**, 141–150 (1954).

Let a star-shaped domain D be represented by the polar equation $r=R(\theta)$, the pole being (h, k). We consider for any real α

$$R_\alpha=\min_{\max}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\{R(\theta)\}^\alpha d\theta\right\}^{1/\alpha}$$

for all (h, k), according as $\alpha\geqq 2$ or $\alpha\leqq 2$. [For $\alpha=2$ the quantity in question is independent of (h, k).] The purpose of the authors is to estimate the lowest eigenvalue λ of D by those of the circle of equal R_α, that is, by the quantities $\lambda_\alpha=j_{01}/R_\alpha$ where j_{01} is the first positive root of $J_0(x)$. It follows trivially that $\lambda\geqq\lambda_\alpha$ provided $2\leqq\alpha\leqq\infty$. The aim is to establish some $\alpha=\alpha'<2$ for which the same inequality should hold for any domain D. For nearly circular domains it is shown that $\alpha=4-j^3_{01}=-1.779\cdots$ is such a value. The special cases of the rectangle and of the circle are considered.
G. Szegö (Stanford, Calif.).

***Purday, H. F. P. Linear equations in applied mechanics.** Oliver and Boyd, Edinburgh-London; Interscience Publishers, Inc., New York, 1954. xiv+240 pp. $3.50.

The object of this book is to help readers with an elementary knowledge of the calculus to get acquainted with the easier aspects of linear algebraic equations, difference equations, ordinary and partial differential equations, and integral equations as well as the associated ideas of matrices, determinants, invariants, vectors, tensors, conjugate functions, orthogonal functions, series etc. (author's preface). The theory is illustrated by applications to continuous beams, oscillators, heat flow, elasticity, hydrodynamics, vibrations of strings and membranes. The book is particularly suited to engineers, a feeling of reality being maintained by numerous numerical solutions worked out in detail. The inclusion of a chapter on linear integral equations, applied to continuous beams, is a novel and useful feature in a book of this character. *J. L. Synge* (Dublin).

***Carrier, G. F. Boundary layer problems in applied mechanics.** Advances in Applied Mechanics, vol. 3, pp. 1–19. Academic Press Inc., New York, N. Y., 1953. $9.00.

It is suggested that the ideas used in boundary-layer theory in hydrodynamics may carry over effectively into other fields as well. Specifically, suppose a regime is governed by a differential equation (ordinary or partial) in which the coefficient ϵ of the most highly differentiated term is very small. The idea is that the equation obtained by setting $\epsilon=0$ will give a solution approximately correct throughout most of the domain of interest, except near the boundaries. Near the boundaries, on the other hand, new variables can be introduced which differ from the old ones because of multiplicative factors which are powers of ϵ. These powers may be determined in such a way that the most highly differentiated term, previously mentioned, now becomes a leading term. Neglecting quantities of lower order, a new differential equation is obtained which is simpler than the

original one. If this has solutions which become small rapidly as one leaves the boundary, a satisfactory boundary-layer theory, whose application is economical from the standpoint of labor, is the result.

These ideas are applied to an ocean-current problem, a heat-flow problem encountered in the calendering process, a convection problem involving a gas whose boundary is maintained at a spacially variable but time-wise constant temperature, and to Van der Pol's equation for the case of large nonlinearity. *E. Pinney* (Berkeley, Calif.).

Miranda, Carlo. Sull'integrazione delle forme differenziali esterne. Ricerche Mat. **2** (1953), 151–182 (1954).

Soit, dans un domaine de R^n à frontière assez régulière, une forme différentielle F_{p-1} de degré $p-1$, de classe C_0, à coefficients höldériens, homologue à 0 (i.e. d'intégrale nulle sur tout $(p-1)$-cycle). Poursuivant un travail antérieur dans lequel il considérait seulement le cas $p=n$ [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **85**, 246–254 (1951); ces Rev. **14**, 46] l'auteur construit explicitement une forme différentielle U_{p-2} telle que $F_{p-1}=dU_{p-2}$ (l'expression de U_{p-2} faisant intervenir la solution d'un problème de Dirichlet pour formes harmoniques). Il résout plus généralement le système: $dU_{p-2}=F_{p-1}$, $dU^*_{p-2}=G_{n-p+2}$, où F et G sont des formes données, de classe C_0, à coefficients höldériens, homologues à 0 ($3\leqq p\leqq n+1$; U^* est l'adjointe de U). *J. Deny* (Strasbourg).

Moisil, Gr. C. Systèmes différentiels adjoints et formules de réciprocité. I, II, III. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3**, 189–207 (1951). (Romanian. Russian and French summaries)

In I, the author discusses systems of partial differential equations adjoint to systems of the form

$$\sum_j P_{ij}\left(\frac{\partial}{\partial x}\right)\varphi_j=0,$$

where $P_{ij}(X)=P_{ij}(X_1,\cdots,X_n)$ is a polynomial in n variables. Examples taken from the theory of mathematical physics are considered. In II, he discusses the integrability of systems of the form

$$\frac{\partial\varphi_i}{\partial x_\mu}+\sum_j a_{\mu ij}\frac{\partial\varphi_j}{\partial y}=0,$$

where the coefficients $a_{\mu ij}$ are constants, with applications to the theory of monogenic functions of a hypercomplex variable. Similarly, in III he discusses systems of the form

$$\frac{\partial\varphi_i}{\partial x_\mu}+\sum_j a_{\mu ij}\frac{\partial\varphi_j}{\partial y}+\sum_j b_{\mu ij}\frac{\partial\varphi_j}{\partial z}=0.$$

E. F. Beckenbach (Los Angeles, Calif.).

Moisil, Gr. C. Systèmes différentiels adjoints et formules de réciprocité. IV. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **4**, 39–51 (1952). (Romanian. Russian and French summaries)

Relative to his earlier analysis (see the preceding review), the author proposes the following problem: For solutions, in a domain D, of a given system of linear partial differential equations

$$\sum_j P_{ij}\left(\frac{\partial}{\partial x}\right)\varphi_j=0$$

with constant coefficients, determine integrals of the form

$$I_i=\int_{V_r}\cdots\int\sum_{j_1\cdots j_{rk}}U_{ij_1\cdots j_{rk}}\left(\frac{\partial}{\partial x}\right)\varphi_k$$

which vanish on every closed r-dimensional variety in D. For examples he considers the Cauchy-Riemann equations and also the Laplace equation in three independent variables; in the latter case he shows that there are no nontrivial solutions for $r=3$, while for $r=2$ there is only the well-known solution expressing the vanishing of the integral of the normal derivative. *E. F. Beckenbach.*

Ionescu-Cazimir, Viorica. Sur les caractéristiques de certains systèmes d'équations aux dérivées partielles. I. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3**, 177–182 (1951). (Romanian. Russian and French summaries)

It is shown that the characteristic curves of systems of partial differential equations of the type

$$\frac{\partial\varphi_i}{\partial x_\mu}=\sum_{j=1}^{m}a_{\mu ij}\frac{\partial\varphi_j}{\partial x_0}$$

($i=1,\cdots,m$; $\varphi=1,\cdots,n$) satisfy equations of Monge type. Examples are given involving monogenic functions of hypercomplex variables. *E. F. Beckenbach.*

Ionescu-Cazimir, Viorica. Sur les caractéristiques de certains systèmes d'équations aux dérivées partielles. II. Acad. Repub. Pop. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. **3**, 183–188 (1951). (Romanian. Russian and French summaries)

The author determines partial differential equations satisfied by the characteristic surfaces of systems of partial differential equations of the type

$$\frac{\partial\varphi_i}{\partial x_\mu}=\sum_{j=1}^{m}a_{\mu ij}\frac{\partial\varphi_j}{\partial x_{n+1}}+\sum_{j=1}^{m}b_{\mu ij}\frac{\partial\varphi_j}{\partial x_{n+2}}$$

($i=1,\cdots,m$; $\varphi=1,\cdots,n$). Examples are given expressing conditions of monogeneity of functions of two complex variables. *E. F. Beckenbach* (Los Angeles, Calif.).

Diaz, J. B., and Martin, M. H. A generalization of Riemann's method for partial differential equations. Ann. Mat. Pura Appl. (4) **36**, 335–359 (1954).

The authors deal with partial differential equations of second order:

$$L(u)=\sum_{i,j=1}^{2}a^{ij}u_{x_i}u_{x_j}+\sum_{i=1}^{2}a^iu_{x_i}=0,$$

$$M(v)=\sum_{i,j=1}^{2}b^{ij}v_{x_i}v_{x_j}+\sum_{i=1}^{2}b^iv_{x_i}=0$$

and seek to determine its coefficients together with two multipliers λ,μ such that $\lambda L(u)+\mu M(v)=\partial A^1/\partial x_1+\partial A^2/\partial x_2$, where A^1, A^2 are bilinear forms in u_{x_1}, u_{x_2}, v_{x_1}, v_{x_2}. Then the line integral $\int(A^2dx_1+A^1dx_2)$ vanishes around closed paths, provided u, v are regular solutions of $L(u)=0$ and $M(v)=0$. This new bilinear divergence identity is used to unify the treatment of Cauchy's problem for hyperbolic equations, the initial-value problem for parabolic equations and the Dirichlet problem for elliptic equations. If the operators L and M satisfy the bilinear divergence identity for all functions u, v (of class C'') in a region R (with the multipliers

$$\lambda=\lambda_iv_{x_i},\quad \mu=\mu_iu_{x_i},\quad \lambda_i=\lambda_i(x_1\cdots x_n),\quad \mu_i=\mu_i(x_1\cdots x_n)),$$

then

$$a^i{}_{jk}+a^j{}_{ik}=2a^{ij}\lambda_k,\quad \frac{\partial a^k{}_{ij}}{\partial x_k}-a^i\lambda_j-b^i\mu_i=0$$

$$a^i{}_{kj}+a^j{}_{ki}=2b^{ij}\mu_k,$$

and vice versa (Theorem 1). Furthermore it is proved (Theorem 2): suppose that the operators $L(u)$ and $M(v)$ in two independent variables, with real coefficients, are associate differential operators with non-trivial multipliers λ and μ. In a region in which $L(u)=0$ is hyperbolic, $L(u)=0$ and $M(v)=0$ must share at least one family of characteristic curves. In a region in which $L(u)=0$ is elliptic, the corresponding coefficients of the second derivative terms in the two operators must be proportional. If the given operator L is actually of second order, the coefficients a^{ij} of the second-order terms do not vanish simultaneously in R. Regarding the matrix

$$N=\begin{Vmatrix} a^{11} & 2a^{12} & a^{22} \\ b^{11} & 2b^{12} & b^{22} \end{Vmatrix}$$

the two cases are considered: when the rank of N is one throughout R and when the rank of N is two throughout R. The first case arises if $L(u)=0$ is elliptic and, if $L(u)=0$ is hyperbolic, when $L(u)=0$, $M(v)=0$ share both families of characteristic curves. The second case is in connection with Laplace's cascade method. Furthermore the authors discuss first three hyperbolic examples

(1) $L(u)=u_{xy}-\dfrac{m}{x-y}(u_x-u_y)=0,$ (Euler-Darboux equation)

(2) $L(u)=u_{tt}+u_t-u_y=0$

(3) $L(u)=u_{xx}-m^2u_{yy}-u_y=0, \quad m=\text{const.},$

second a parabolic example

$$L(u)=u_{xx}-u_y=0 \quad \text{(linear heat flow)}$$

and finally an elliptic example

$$\Delta u=u_{xx}+u_{yy}=0 \quad \text{(Laplace's equation).}$$

M. Pinl (Cologne).

Gagua, M. E. On some questions of uniform approximation of solutions of differential equations of elliptic type. Izvestiya Akad. Nauk SSSR. Ser. Mat. **18**, 177–184 (1954). (Russian)

The author considers solutions of

(1) $$\Delta u+a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}+c(x,y)u=0$$

in a simply connected region T with boundary L. The coefficients a, b, c are assumed to be entire and the mapping function of T is assumed to be continuous on L and of class A_p. Designate by $G(t,\tau,z,\zeta)$ the Riemann's function of (1), and by $\bar{T}$, $\bar{L}$ the reflection of T, L in the real axis. Theorem: Let $u(x,y)$ be a regular solution of (1) in T whose analytic continuation $U(z,\zeta)$, $z=x+iy$, $\zeta=x-iy$, is such that $U(z,0)$ and $U(0,\zeta)$ are continuous in $T+L$ and $\bar{T}+\bar{L}$ respectively. Then $u(x,y)$ is continuous in $T+L$ and can be approximated uniformly there by particular solutions u_{nm} of the form

$$u_{nm}(x,y)=G(z,0,z,\bar{z})P_n(z)-\int_0^z P_n(t)\frac{\partial}{\partial t}G(t,0,z,\bar{z})\,dt$$

$$+G(0,z,z,\bar{z})Q_m(\bar{z})-\int_0^z Q_m(\tau)\frac{\partial}{\partial \tau}G(0,\tau,z,\bar{z})\,d\tau,$$

where P_n and Q_m are polynomials of degree n and m.

P. Davis (Washington, D. C.).

Burgers, J. M. On the coalescence of wave like solutions of a simple non-linear partial differential equation. I, II, III. Nederl. Akad. Wetensch. Proc. Ser. B. **57**, 45–56, 57–66, 67–72 (1954).

The author continues his investigation of the wavelike solutions of the non-linear partial differential equation $v_t+vv_x=\nu v_{xx}$. In this paper, emphasis is on a statistical treatment of the merging of "shock waves" created by impulses introduced at $y=0$ at successive instants. By limiting himself to the case where the impulses follow each other at regular intervals, but have random amplitude distributions, the author is able to find various statistical mean values in terms of the interval T between the successive impulses, the mean strength Ω of these impulses, and their standard deviation σ from this mean value. *C. C. Lin.*

Burgers, J. M. Further statistical problems connected with the solution of a simple non-linear partial differential equation. Nederl. Akad. Wetensch. Proc. Ser. B. **57**, 159–169 (1954).

This paper continues the investigation [see the preceding review] of the statistical properties of the solution of the differential equation $v_t+vv_x=\nu v_{xx}$. Unlike the previous problem, initial-value problems are considered. Again statistical averages are calculated by the explicit use of the solution of the differential equation for small ν. It is pointed out that many other statistical properties of the solution can be treated along similar lines. *C. C. Lin.*

Weinig, F. S. Solutions of heat-conduction problem with the aid of the inverse method. J. Appl. Mech. **20**, 489–496 (1953).

The author considers the equation $\Delta t=0$ with the boundary condition $t+r\partial t/\partial n=T$, where r is a constant and T a prescribed temperature. In the limiting cases $r\to 0$, $r\to\infty$ any potential line or streamline can be chosen as a boundary. The author shows that also in the case of finite values of r boundaries can be found such that the given boundary condition is fulfilled. A number of special cases are given from which we mention the case in which the boundary is formed by radial fins on a cylinder. *H. A. Lauwerier.*

Jörgens, Konrad. Über die Lösungen der Differentialgleichung $rt-s^2=1$. Math. Ann. **127**, 130–134 (1954).

Let $z=z(x,y)$ be a solution of the differential equation $rt-s^2=1$ in the circular disc $(x-x_0)^2+(y-y_0)^2<R^2$, where the partial derivatives $r=\partial^2z/\partial x^2$, $s=\partial^2z/\partial x\partial y$, $t=\partial^2z/\partial y^2$ are continuous in the indicated disc. On denoting by r_0, s_0, t_0 the values of r, s, t at the center (x_0,y_0) of the disc, the main result of the paper is stated as follows: $z(x,y)$ is analytic in the disc, and the third partial derivatives of $z(x,y)$ satisfy the inequalities

$$[r_x^2+r_y^2]_{x=x_0,\,y=y_0}<24(r_0+t_0)^2r_0^2/R^2,$$
$$[t_x^2+t_y^2]_{x=x_0,\,y=y_0}<(r_0+t_0)^2r_0^2R^2.$$

As an immediate corollary, there follows the theorem: if $z(x,y)$ possesses continuous partial derivatives of the first and second order in the whole (x,y)-plane and satisfies there the differential equation $rt-s^2=1$, then $z(x,y)$ reduces to a polynomial of second degree. The paper contains several interesting comments upon the relationships between these results and previous work by E. Heinz, L. Bers, S. Bernstein, E. J. Mickle, and the reviewer. *T. Radó.*

Difference Equations, Special Functional Equations

Have

***Bellman, Richard. A survey of the mathematical theory of time-lag, retarded control, and hereditary processes. With the assistance of John M. Danskin, Jr.** The Rand Corporation, Santa Monica, Calif., 1954. xi+107 pp.

In an extensive variety of physical phenomena (including servomechanisms and some other control devices) it is essential to know at least a part of their past history. One thus considers systems

$$(1)\quad \frac{dy_i(t)}{dt}=G_i[y_1(t),\cdots,y_N(t)]+\int_{-\infty}^{t}F_i[y_1(s),\cdots,y_N(s)]dK_i(s,t),$$

which sometimes can be approximated by d.-d. (differential-difference) systems

$$(2)\quad \frac{dy_i(t)}{dt}=F_i[y_1(t),\cdots,y_N(t),y_1(t-\tau),\cdots,y_N(t-\tau)]$$

(τ being the time lag and $y_i(t)=\phi_i(t)$ for $0\leqq t\leqq\tau$, where the $\phi_i(t)$ are known). The author is largely concerned with the system (2) and he gives a systematic presentation of the theory of such systems in a form facilitating applications to problems such as those of automatic pilotless control, of the prediction of the course of an airplane (when velocity is supersonic) and so forth. An exposition of the Laplace transform, its inversion and convolution, is followed by a study of linear d.-d. equations with constant coefficients. Some existence and uniqueness theorems are obtained for the "renewal" equation $u(t)=f(t)+\int_0^t u(t-t_1)dG(t_1)$, when $dG=gdt_1$. This is followed by a study of the asymptotic properties of the solution, connecting with the Tauberian theorems of Hardy and Littlewood and Ikehara. An extension to systems of renewal equations (connecting with positive operators) affords opportunities for further investigations of notable mathematical interest. A study of the zeros of exponential polynomials is followed by a stability theory for d.-d. equations (analogues of the classical results of Poincaré and Liapounoff); this connects with the work of E. M. Wright. A study is given of the general control problem, related to d.-d. equations; this is, in part, based on the earlier work of Minorsky. Applications are given to mathematical economics (the macrodynamic model of Kalecki). There is an extensive bibliography. The survey is a skillful and useful exposition of a field, in which there still remains considerable room for further research.

W. J. Trjitzinsky (Urbana, Ill.).

Čermák, J. On a new method of solving homogeneous systems of linear difference equations with constant coefficients. Ann. Polon. Math. 1, 195–202 (1954).

Let us be given a set of n ordinary homogeneous linear difference equations of the first order with constant coefficients and involving n dependent variables. As the author remarks, the general solution for such a set, barring algebraic difficulties, has been known a long time. The present paper adds a novel point of view. A fundamental system of solutions is obtained by means of what is called reduced normal systems of vectors relative to the characteristic roots of the matrix of the coefficients. Algebraic difficulties remain.

T. Fort (Athens, Ga.).

Franklin, Joel. On the existence of solutions of systems of functional differential equations. Proc. Amer. Math. Soc. 5, 363–369 (1954).

The system

$$y_\alpha'(x)=f_\alpha(x,y_\beta(u_k(x)))\quad(\alpha,\beta=1,\cdots,n;\ k=1,\cdots,m)$$

is considered on $0\leqq x\leqq x_1$ under suitable initial conditions, and with continuous u_k such that $u_k(x)\leqq x$; for fixed x, $y_\beta(u_k(x))$ denotes a system of mn numbers, that with indices β_0 and k_0 being $y_{\beta_0}(u_{k_0}(x))$. An existence and uniqueness theorem similar to Picard's is proved with f_α Lipschitz-continuous, and an existence theorem similar to Peano's is proved with f_α continuous. *F. A. Ficken.*

Bajraktarević, Mahmud. Sur les solutions d'une équation fonctionnelle. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 8, 297–300 (1953). (Serbo-Croatian. French summary)

Anastassiadis [Bull. Sci. Math. (2) 76, 148–160 (1952); these Rev. 14, 761] has determined explicitly the unique positive function $f(x)$, $x>0$, which is semi-decreasing 1 or semi-convex 1, and which satisfies the functional equation $f(x)f(x+1)=1/P(x)$, where $P(x)=(x+\rho_1)\cdots(x+\rho_k)$ ($\rho_1,\cdots,\rho_k\geqq 0$). The result is now strengthened by dropping the restrictions $f(x)>0$ and $x>0$. *E. F. Beckenbach.*

Hosszu, M. On the functional equation of autodistributivity. Publ. Math. Debrecen 3 (1953), 83–86 (1954).

It is shown that if the function $M(x,y)$ is strictly monotonic and once differentiable in each variable, and satisfies the functional equations

$$(*)\qquad \begin{aligned} M[M(x,y),z]&=M[M(x,z),M(y,z)],\\ M[z,M(x,y)]&=M[M(z,x),M(z,y)],\end{aligned}$$

then $M(x,y)$ can be expressed in the form

$$M(x,y)=f^{-1}[pf(x)+qf(y)],$$

where $f(t)$ is a strictly monotonic differentiable function and p and q are constants satisfying $p+q=1$, and conversely. The author remarks that in oral communication J. Aczel established the same result assuming second-order derivatives but omitting one of (*). *E. F. Beckenbach.*

Hosszu, M. A generalization of the functional equation of bisymmetry. Studia Math. 14 (1953), 100–106 (1954).

The author determines the form of each of the six functions appearing in the generalized bisymmetric functional equation

$$F[G(x,y),H(u,v)]=f[g(x,u),h(y,v)]$$

on the assumption that the functions are differentiable and strictly monotonic. The solutions turn out to be interrelated functions of the form $z=Z[X(x)+Y(y)]$.

E. F. Beckenbach (Los Angeles, Calif.).

Aczél, J. Grundriss einer allgemeinen Behandlung von einigen Funktionalgleichungstypen. Publ. Math. Debrecen 3 (1953), 119–132 (1954).

The author treats certain classes of functional equations, most of which include previously studied functional equations as special cases. Thus (*) $f(x+y)=F[f(x),f(y)]$ includes the Cauchy equation $f(x+y)=f(x)+f(y)$, and $f[\frac{1}{2}(x+y)]=F[f(x),f(y)]$ includes the Jensen equation $f[\frac{1}{2}(x+y)]=\frac{1}{2}[f(x)+f(y)]$, etc. Methods of solution are discussed, and in most cases existence and uniqueness proofs are given. Thus it is shown that (*) has a continuous and

strictly monotonic solution if and only if $F(x, y)$ is continuous and strictly monotonic in both variables and satisfies the associativity condition $F[F(x, y), z] = F[x, F(y, z)]$; for a given (x_0, y_0) there can be at most one solution $y = f(x)$ satisfying $y_0 = f(x_0)$ because if $f(x)$ is a solution then the totality of solutions is given by $g(x) = f(ax)$.

E. F. Beckenbach (Los Angeles, Calif.).

Functional Analysis

Dieudonné, J. A. Recent developments in the theory of locally convex vector spaces. Bull. Amer. Math. Soc. **59**, 495–512 (1953).

An instructive survey of the general theory of locally convex vector spaces (l.c.v.s.) including, besides more classical questions, topics such as inductive and projective limits and various tensor products. Different important classes of l.c.v.s., their properties and interrelations are considered (e.g., "espaces tonnelés", semi-reflexive spaces, bornological spaces, and many others) and some unsolved problems are stated. The author stresses the idea of classifying l.c.v.s. "according to their behavior with regard to the validity of the main theorems of functional analysis."

M. Katětov (Prague).

Grothendieck, Alexandre. Sur certains espaces de fonctions holomorphes. II. J. Reine Angew. Math. **192**, 77–95 (1953).

This is the second installment of a long paper [for part I see same J. **192**, 35–64 (1953); these Rev. **15**, 438], and the notations and definitions of the earlier review will not be repeated here. Much of this second installment is concerned with the concept of a nuclear space, a concept which is introduced with the aid of the notion of a topological tensor product of two locally convex linear topological spaces. The author has dealt with tensor products and nuclear spaces at length in another memoir, not yet published. Some of these unpublished results are used in the present paper. Indeed, the author says that a good part of the results here developed for spaces of analytic functions can be formulated abstractly in the theory of nuclear spaces. Until this other memoir is available, readers of the present paper will find some help in the paper of Dieudonné reviewed above.

The starting point is the observation that the space $P(\Omega_1, E)$ is a topological tensor product of $P(\Omega_1)$ and E. This means, in the present instance, that if we write $\varphi \otimes a$ for the element of $P(\Omega_1, E)$ whose value at z is $\varphi(z)a$ (where $\varphi \in P(\Omega_1)$ and $a \in E$), then there is a natural one-one mapping of $P(\Omega_1, E)'$ (prime denoting the dual space) onto $\mathfrak{B}(P(\Omega_1) \times E)$ (the space of continuous bilinear functionals on $P(\Omega_1) \times E$) defined by $u \to \tilde{u}$, where $\tilde{u}(\varphi, a) = u(\varphi \otimes a)$. This mapping, moreover, carries equicontinuous sets into equicontinuous sets. Equicontinuity of a set refers to the fact that the elements of the set are linear (or bilinear) functions, and that as such they are equicontinuous at the origin. It follows that if $\Omega_1 = O$ is a nonempty proper subset of the Riemann sphere, and if E is a complete LCS, $P(O)$ is a nuclear space. This means, in the present instance, that there is an isomorphism between $P(O, E)$ and the space $\mathfrak{L}(E', P(O))$ of weakly continuous linear mappings of E' into $P(O)$, the topology of $\mathfrak{L}(E', P(O))$ being that of uniform convergence on the equicontinuous sets in E'. This isomorphism is the canonical one which makes $F \to T$ if $F \in P(O, E)$, $T \in \mathfrak{L}(E', P(O))$, and T is defined by

$$Ta' = \varphi(a' \in E', \varphi \in P(O)),$$

where $\varphi(z) = \langle F(z), a' \rangle$.

The property of being nuclear is possessed by the space $H(O, E)$ of analytic functions, with values in E, defined on any analytic manifold O of one or more complex dimensions. Several results about $H(O, E)$ are obtained by considering $H(O, E)$ as a subspace of the space of infinitely differentiable functions on O to E. This latter space is susceptible to investigation by the methods of L. Schwartz. One of the consequences of being nuclear is that bounded sets in $H(O, E)$ are relatively compact (i.e., have compact closures) if and only if the same is true of E. Also, a set A in $H(O, E)$ is relatively weakly compact if and only if A is bounded and, for each $z \in O$, the set of values $\{f(z)\}$, $f \in A$, is relatively weakly compact in E. As a result, $H(O, E)$ is semi-reflexive if and only if E is semi-reflexive (for the definition of this term see the cited paper by Dieudonné). An LCS M is called distinguished if every weakly bounded subset of the bidual M'' is in the weak closure of some bounded subset of M (under the canonical imbedding of M in M''). If E is metrizable and complete, $H(O, E)$ is distinguished if and only if E is.

The last section of the paper deals again with the spaces $P(\Omega_1, E)$, Ω_1 being a non-void proper subset of the extended complex plane. Conditions are given sufficient to insure that $P(\Omega_1, E)$ is a t-space (French tonnelé). One such condition is that E be metrizable and complete. Other theorems result from considering the case when E is a strongly closed subspace of the strong dual of a complete metrizable space. For example: If E is a reflexive Banach space, $P(\Omega_1, E)$ and $P(\Omega_2, E')$ are reflexive, and strong duals of each other (Ω_2 the complement of Ω_1). Further, if E is any Banach space, or if E is metrizable and complete and Ω_1 is open, then, giving E' its strong topology, the topology of $P(\Omega_2, E')$ is identical with the strong topology of the dual of $P(\Omega_1, E)$. Finally, conditions on E are given which insure that $P(\Omega_1, E)$ is complete or that bounded sets in $P(\Omega_1, E)$ are relatively compact. These conditions are satisfied in particular if E is the complex plane. Examples are given to show that the restrictions on E are not superfluous. *A. E. Taylor.*

Alexiewicz, A. A theorem on the structure of linear operations. Studia Math. **14** (1953), 1–12 (1954).

Let X be a separable F space, Y an F space consisting of functions from a measure space $(T, \mathfrak{S}, \mu)$ to a linear space L. The space Y is structured by the assumptions $(y_e(t) = y(t)\chi_e$ where χ_e is the characteristic function of the set $e \in \mathfrak{S})$: (1) $y \in Y$ implies $y_e \in Y$ and $|y_e| \leqq |y|$; (2) $\mu(e_n) \to 0$ implies $|y_{e_n}| \to 0$; (3) for some analytic and linear set $B \subset Y$, $y \in B$ implies $y_e \in B$ (hence y_e, $y_h \in B$ implies $y_{e \cup h} \in B$); (4) $y_{e_n} \in B$ and $e = \bigcup_1^\infty e_n$ implies $y_e \in B$. Theorem: Let U be a linear operation from X to Y. Then there is a measurable decomposition of $T = e \cup h$ and a residual set $R \subset X$ such that (a) for $x \in X$, $\epsilon > 0$, there is a measurable e' such that $\mu(e - e') < \epsilon$ and $(U(x))_{e'} \in B$, (b) for $x \in R$ and $h' \subset h(\mu(h') > 0)$,

$$(U(x))_{h'} \text{ non-} \in B.$$

Applications to function spaces are given. *B. Gelbaum.*

Albrycht, J. On a theorem of Saks for abstract polynomials. Studia Math. **14** (1953), 79–81 (1954).

The author generalizes a result of Alexiewicz [Studia Math. **13**, 18–29 (1953); these Rev. **14**, 1070] by strengthening a category-type lemma of the original paper. The final

result reads: Let $U(x, t)$ be a mapping of a separable F space X to an F-space Y such that (a) $U(x, t)$ is Bochner measurable for each x (note: $t \in T$ where $(T, \mathfrak{S}, \mu)$ is a measure space in the Halmos notation), (b) $U(x, t)$ is a polynomial of degree m in x for fixed t, (c) $x \to x_0$ implies $U(x, t) \to (x_0, t)$ in measure. Then if R is a linear Borel measurable set in F, there is a decomposition of $T = A \cup B$ into two measurable sets and a residual set $Z \subset X$ such that (1) $U(x, t) \in R$ for all $x \in X$, a.e. in A, (2) $U(x, t)$ non-$\in R$ for $x \in Z$ a.e. in B.

B. Gelbaum (Minneapolis, Minn.).

Fleischer, Isidore. Sur les espaces normés non-archimédiens. Nederl. Akad. Wetensch. Proc. Ser. A. **57** = Indagationes Math. **16**, 165–168 (1954).

This paper settles the structure of normed left vector spaces E over noncommutative fields K complete with respect to a discrete non-Archimedean valuation such that E also possesses a non-Archimedean discrete valuation (i.e., the strong triangle inequality must hold in E). The principal theorem asserts that E contains densely an isometric copy of a space $N(a_R)$ to be defined presently, and that if E is complete it is isometric with the completion $C(a_R)$ of $N(a_R)$. Here $N(a)$ is a left K-vector space with a Hamel basis $\{x_i\}$ of cardinality a normed as follows: $\|\sum_{i=1}^{n} \lambda_i x_{i_k}\| = \max |\lambda_k|$. The proof is based on the observation that if we set $M = [x \in E; \|x\| \leqq 1]$, $M' = [x \in E; \|x\| < 1]$, then $\bar{M}_R = M$ modulo M' is a vector space over the residue class field $\bar{K}$ of K and hence has a Hamel basis whose antecedents in M furnish the desired basis for E. [If E is separable, this theorem was proved by Monna, same Proc. **51**, 197–210 (1948); these Rev. **9**, 517.] This theorem implies the existence of "supplements", where two subspaces V and W of E are called supplementary if $x \in V$, $y \in W$ implies

$$\|x+y\| = \max(\|x\|, \|y\|)$$

and if E is the direct sum of V and W; it also implies the analog of the Hahn-Banach Theorem [see Monna, ibid. **49**, 1134–1141 (1946); these Rev. **9**, 43; Cohen, ibid. **51**, 693–698 (1948); these Rev. **10**, 48; see also Ono, J. Math. Soc. Japan **5**, 1–5 (1953); these Rev. **15**, 717, and the paper reviewed below]. The dual E^* of E turns out to be the (complete) vector space of all bounded K-valued functions on a set of cardinality a_R with sup norm. If E_3 is the K-space of all continuous K-linear mappings of E_1 into E_2, then $a_{R_3} = b_2^{b_1}$ where b_i is the cardinality of $\bar{M}_{R_i}$. This shows that no infinite-dimensional E can be reflexive. A final section deals with the problem of the existence of a non-Archimedean norm in a topological vector space E. If $A = [\lambda \in K; |\lambda| \leqq 1]$ and if the topology of E can be derived from an A-module M in E by the formation of all sets λM $(\lambda \neq 0)$, then E satisfies the strong triangle inequality. There do exist normed K-vector spaces whose norm is not equivalent to another norm satisfying the strong triangle inequality (e.g., l_1 over K).

G. K. Kalisch (Minneapolis, Minn.).

Ono, Takashi. A generalization of the Hahn-Banach theorem. Nagoya Math. J. **6**, 171–176 (1953).

The author is concerned with noncommutative fields k and normed spaces S over them such that either (a) k is the field of the real or complex numbers or the quaternions, or (b) k is complete in a discrete non-Archimedean valuation and S is non-Archimedean. He further considers σ-linear functionals f (i.e., $f(\alpha x+\beta y) = \alpha^\sigma f(x) + \beta^\sigma f(y)$ for all α and β in k and x and y in S) for valuation-preserving automorphisms σ of k. Under these circumstances, the Hahn-Banach extension theorem is true for σ-linear functionals [if σ is the identity see G. A. Suhomlinov, Mat. Sbornik N.S. **3** (**45**), 353–358 (1938) in case (a); for case (b) see the author's paper in J. Math. Soc. Japan **5**, 1–5 (1953); these Rev. **15**, 717, and papers cited there; see also the paper reviewed above]. The usual corollaries on the existence of such σ-linear functionals hold [see, e.g., S. Banach, Théorie des opérations linéaires, Warszawa-Lwów, 1932, pp. 55, 57]. The foregoing is then applied to the pertinent generalization of the theorem characterizing normed spaces S through their lattice of closed subspaces [see Mackey, Ann. of Math. (2) **43**, 244–260 (1942); these Rev. **4**, 12]; this theorem contains a hypothesis on the valuation-preserving character of a certain automorphism of k; the author shows that all automorphisms of finite-dimensional division algebras over p-adic fields R_p are valuation-preserving and that R_p has only the identity automorphism [see, e.g., Chevalley, J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. **2**, 365–476 (1933)]. As a second application, the author proves that if k is commutative, the ring $A(S)$ of all continuous k-endomorphisms of S is semisimple. Finally, the author proves the pertinent generalization of Eidelheit's Theorem [Studia Math. **9**, 97–105 (1940); these Rev. **2**, 224] that S is characterized by $A(S)$ and that all automorphisms of $A(S)$ over k are inner.

G. K. Kalisch (Minneapolis, Minn.).

Barry, John Y. On the convergence of ordered sets of projections. Proc. Amer. Math. Soc. **5**, 313–314 (1954).

The author considers the problem of whether a limit projection exists to a uniformly bounded naturally ordered set of projections in a Banach space X. He shows that if $\{E_\alpha\}$ is such a set of projections which has for each x in X a weak x-cluster point, then the strong limit of $\{E_\alpha\}$ exists. Definition: y_α is a weak x-cluster point of $\{E_\alpha\}$ if for each α, y_α belongs to the weak closure of the set of all $E_\beta(x)$ with $\beta \geqq \alpha$. A refinement of this result leads the author to obtain the known result [Lorch, Trans. Amer. Math. Soc. **45**, 217–234 (1939)] that if X is reflexive, the limit of $\{E_\alpha\}$ always exists.

E. R. Lorch (New York, N. Y.).

Tatarkiewicz, Krzysztof. Une théorie généralisée de la meilleure approximation. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **6** (1952), 31–46 (1954). (Polish and Russian summaries)

Suppose E is a normed linear space and F is a finite-dimensional linear subspace of E. For each point p of E let $\mu(p) = \inf_{x \in F} \|p-x\|$ and $A(p) = \{y : y \in F \text{ and } \|p-y\| = \mu(p)\}$. It is observed that μ is continuous, that $A(p)$ is always non-empty, compact, and convex, that A is everywhere upper semi-continuous and is lower semi-continuous at each point p for which $A(p)$ is one-pointed. Examples show that lower semi-continuity may fail without this additional restriction. A characterization is given of subspaces F such that $A(p)$ is one-pointed for all $p \in E$.

V. L. Klee.

Fenyő, I. Über die Lösung der im Banachschen Raume definierten nichtlinearen Gleichungen. Acta Math. Acad. Sci. Hungar. **5**, 85–93 (1954). (Russian summary)

The author proves: If X and Y are Banach spaces and if f: $X \to Y$ is such that (1) f has a Fréchet derivative $f'(x)$ near x_0, and $f'(x_0)$ has an inverse operator with $\|f'(x_0)^{-1}\| \leqq B$; and (2) $\|f'(x) - f'(x_0)\| \leqq C\|x - x_0\|$ for $\|x - x_0\| \leqq r < (3BC)^{-1}$, then it follows that if $\|f(x_0)\| \leqq r(2-3BC)(11B)^{-1}$ the equation $f(x) = 0$ has a unique solution x with $\|x - x_0\| < r$ which is the limit of Newton's iterative process

$$x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n), \quad n = 0, 1, 2, \cdots.$$

A similar result is established for the modified process $x_{n+1}=x_n-[f'(x_0)]^{-1}f(x_n)$. These results are related to theorems of Kantorovič [Doklady Akad. Nauk SSSR (N.S.) 59, 1237–1240 (1948); Uspehi Matem. Nauk (N.S.) 3, no. 6(28), 89–185 (1948); Trudy Mat. Inst. Steklov. 28, 104–144 (1949); these Rev. 9, 537; 10, 380; 12, 419] and Mysovskih [Doklady Akad. Nauk SSSR (N.S.) 70, 565–568 (1950); Trudy Mat. Inst. Steklov. 28, 145–147 (1949); these Rev. 11, 601; 12, 419] who imposed boundedness restrictions on the second Fréchet derivative, and to M. L. Stein [Proc. Amer. Math. Soc. 3, 858–863 (1952); these Rev. 14, 1094] who treated analytic mappings between complex Banach spaces. *R. G. Bartle.*

Kato, Tosio. On the semi-groups generated by Kolmogoroff's differential equations. J. Math. Soc. Japan 6, 1–15 (1954).

Let $A=(a_{jk})$ be a given Kolmogoroff matrix,

$$a_{jk}\geqq 0,\quad j\neq k,\quad -a_{jj}=a_j\geqq 0,\quad \sum_k a_{jk}=0,$$

with the corresponding differential equations

$$y'_{jk}(t)=\sum_{m=1}^{\infty}a_{jm}y_{mk}(t),\quad z'_{jk}(t)=\sum_{m=1}^{\infty}z_{jm}(t)a_{mk},$$

and the initial conditions

$$\lim_{t\to 0+}y_{jk}(t)=\lim_{t\to 0+}z_{jk}(t)=\delta_{jk}.$$

The author shows the existence of at least one matrix $P(t)=(p_{jk}(t))$, satisfying both differential equations and the initial conditions, such that $0\leqq p_{jk}(t)$ and $\sum_k p_{jk}(t)\leqq 1$, and $P(t)$ has the semi-group property $P(t+s)=P(t)P(s)$. This result, which goes back to W. Feller [Trans. Amer. Math. Soc. 48, 488–515 (1940); these Rev. 2, 101], is here proved by constructing a semi-group generated by a contraction of A. Here A is considered as an operator in the space (l), the set of vectors $x=(\xi_j)$ with $\|x\|=\sum|\xi_j|$. Here $Ax=y=(\eta_k)$ where $\eta_k=\sum_j \xi_j a_{jk}$ and the domain $\mathfrak{D}[A]$ of A is taken as the set of x's for which the η-series are absolutely convergent and $\sum|\eta_k|<\infty$. The domain $\mathfrak{D}[A]$ contains all unit vectors (δ_{jk}). Let $\mathfrak{D}_0$ be the subspace of (l) spanned by the (δ_{jk}) and let A_0 be the contraction of A to $\mathfrak{D}_0$ so that $\mathfrak{D}[A_0]=\mathfrak{D}_0$. Now write $A=-H+K$ where $-H$ is the diagonal part of A and K is the rest. The domain $\mathfrak{D}$ of the operator H is the set of all x such that $\sum a_k|\xi_k|<\infty$. Here $\mathfrak{D}_0\subset\mathfrak{D}$. For $\lambda>0$, the resolvent $R(\lambda;-H)$ exists as a bounded linear positive operator and $\lambda\|R(\lambda;-H)\|\leqq 1$. The operator H itself is closed linear positive and $\mathfrak{D}$ is dense in (l); K is defined as a linear positive operator in $\mathfrak{D}$ by $Kx=(\eta_k)$ where $\eta_k=\sum_{j\neq k}\xi_j a_{jk}$ and $\|Kx\|\leqq\|Hx\|$ with equality for positive elements of D. Further, $B(\lambda)=KR(\lambda;-H)$ is a bounded linear positive operator, $\|B(\lambda)\|\leqq 1$ and $0\leqq B(\mu)\leqq B(\lambda)$ for $0<\lambda<\mu$. Set $G_r=-H+rK$, $0<r<1$, with $\mathfrak{D}[G_r]=\mathfrak{D}$. Then $R(\lambda;G_r)$ exists for $\lambda>0$ as a linear bounded positive operator with $\lambda\|R(\lambda;G_r)\|\leqq 1$ so that G_r is the infinitesimal generator of a positive contraction semi-group $P_r(t)$ by the Hille-Yosida theorem. Here

$$R(\lambda;G_r)=R(\lambda;-H)\sum_n r^n[B(\lambda)]^n$$

is an increasing function of r so that $P_r(t)$ has the same property. This implies that $P_r(t)$ for $r\to 1-$ converges strongly to a positive contraction semi-group operator $P(t)$. The infinitesimal generator G of $P(t)$ is a contraction of A and a closed extension of $-H+K$ and, a fortiori, of A_0. If there exists any other positive semi-group $P_1(t)$ generated by a closed extension of A_0, then $P(t)\leqq P_1(t)$. In this sense $P(t)$ is the minimal solution of the problem (it is not obvious that $P(t)$ coincides with Feller's solution which has a differently defined minimal property). The matrix $P(t)=(p_{jk}(t))$ satisfies the differential equations and the inequalities stated above. Here $\sum_k p_{jk}(t)=1$ for all j ($P(t)$ is a transition operator) if and only if $[B(\lambda)]^n\to 0$ strongly as $n\to\infty$ for each $\lambda>0$. An equivalent condition is that the range of $\lambda I-A_0$ be dense in (l). Illustrations are given to bounded matrices, triangular matrices, the "birth process" and the "birth-and-death process". *E. Hille* (New Haven, Conn.).

Massera, Jose L. Conditional stability of homeomorphisms. Bol. Fac. Ingen. Montevideo 4, 455–486 (1952). (Spanish. English summary)

Let the Banach space $Z=X\times Y$ be the topological product of Banach spaces X and Y. Let T be a homeomorphism of a neighborhood of the origin 0 of Z onto a neighborhood of 0 leaving 0 fixed. A set E is called a set of stability if, given any neighborhood W of 0, there exists a neighborhood W' of 0 such that, for each $n\geqq 0$, $T^n(E\cap W')\subset W$; E is called a set of instability if it is a set of stability for T^{-1}. The object is to connect the existence and properties of sets of stability with regularity properties of T, which vary from Fréchet differentiability to analyticity. In most theorems either X or Y is finite-dimensional or T^{-1} is differentiable at 0. Under detailed hypotheses detailed results are obtained, for example, on asymptotic stability, on maximality (in various senses) of sets of stability, and on transformations $x\to y$ (on sets $\|x\|\leqq a$ into Y) that are invariant under T and have graphs that are sets of stability.

F. A. Ficken (Knoxville, Tenn.).

Ruston, A. F. Operators with a Fredholm theory. J. London Math. Soc. 29, 318–326 (1954).

Define a Riesz operator on a complex Banach space $\mathfrak{B}$ as one which has the three properties: (1) for every λ the solutions of $(I-\lambda K)^n x=0$ form a finite-dimensional subspace of $\mathfrak{B}$ independent of n for n sufficiently large; (2) $(I-\lambda K)^n\mathfrak{B}$ is a closed subspace of $\mathfrak{B}$ independent of n for n sufficiently large; (3) the characteristic values of K have no finite limiting point. The author shows that this is equivalent to being a Fredholm operator, one for which there exist the permanently convergent power series, generalizations of Fredholm determinants as defined in previous papers [Proc. London Math. Soc. (3) 1, 327–384 (1951); 3, 368–377 (1953); these Rev. 13, 468; 15, 134]. It is also shown that these properties in turn are equivalent to an asymptotic quasi-compact property, viz., if $\kappa(T)=\inf\|T-C\|$ for all C compact (completely continuous) operators on $\mathfrak{B}$, then $\lim_n[\kappa(K^n)]^{1/n}=0$. The equivalence still holds if in the definition of $\kappa(T)$ the totality of compact operators C is replaced by the totality of finite-dimensional operators. *T. H. Hildebrandt.*

Taldykin, A. T. On the existence of characteristic values and on the completeness of a system of characteristic elements for some linear operators. Mat. Sbornik N.S. 34(76), 201–212 (1954). (Russian)

Conditions are given for a bounded operator in a Hilbert space H to have a characteristic value inside its Fredholm circle, we shall say, to be CV. The radius of the Fredholm circle is denoted by r_A for an operator A. A necessary and sufficient condition for A to be CV is that

$$\limsup\{\|A^n f\|/\|f\|\}>1/r_A \tag{1}$$

for at least one f. If the resolvent, A_λ, defined by

$A_\lambda(E-\lambda A)=A$, is expanded in the form

$$A_\lambda=\sum_{k=-m}^{\infty}(\lambda-\lambda_0)^k a_k$$

near a pole λ_0, then, for any h such that $a_{-m}h=0$, $a_{-m}h$ is a characteristic element of A. The elements $a_{-k}h$ are called adjoined elements. If $m>1$, $a_{-m}h$ is orthogonal to all characteristic elements of A.

If C is a bounded operator with a characteristic value μ_0 inside its Fredholm circle and if there is an f not orthogonal to the characteristic and adjoined elements of C for μ_0 and such that, for a bounded A,

$$\|A^n f\| \geqq m_f^n \|C^n f\|, \tag{2}$$

where $r_A m_f \geqq r_C$, then A has a characteristic value λ_0 inside its Fredholm circle and $|\lambda_0| \leqq |\mu_0|/m_f$. Let B commute with A. Then if $C=AB$ is CV, and $r_C\|B\| \leqq r_A$, then A is CV. If $\|B\| \leqq M\|A\|$ and if $A+B$ is CV and $r_{A+B}(1+M) \leqq r_A$, then A is CV.

For the system of characteristic and adjoined values of A^* belonging to characteristic values in its Fredholm circle to form a complete system it is necessary and sufficient that (1) holds for any f in H. If (2) holds and the system for C^* is complete, so is that for A^*. If B is bounded and commutes with A, if $r_C\|B\| \leqq r_A$, where $C=AB$, then if C has a complete system of characteristic and adjoined elements, so has A. The same holds if $A+B$ has a complete system and the inequalities at the end of the previous paragraph hold.

J. L. B. Cooper (Johannesburg).

Krasnosel'skiĭ, M. A., and Ladyženskiĭ, L. A. The structure of the spectrum of positive nonhomogeneous operators. Trudy Moskov. Mat. Obšč. **3**, 321–346 (1954). (Russian)

The authors discuss the existence of positive eigenvalues for a not necessarily linear operator A which is positive with respect to a partial ordering produced by a cone K in a Banach space E. A basic result is: If A is positive, and completely continuous and if G is a bounded open set containing the 0 element of E, then $\inf_{S\cap K}|Ax|>0$ (S the boundary of G) implies A has a positive eigenvalue, and the associated eigenvector lies on $S\cap K$.

An operator A is said to have B as a minorant if for all positive x, $Ax-Bx$ is positive. Theorem: If A is completely continuous and has a positive minorant then the set of positive eigenvectors connects 0 to ∞ (i.e., meets the boundary of every bounded open set containing 0.) An operator A is called x_0-bounded if for each positive element y, there is a natural number p and two positive numbers and β such that $\alpha x_0 \leqslant A_y^p \leqslant \beta x_0$ ($\leqslant$, the partial ordering). Theorem: If A is completely continuous and x_0-bounded, then A has one and only one positive eigenvector. Let A be positive, P and Q positive, completely continuous and x_0 bounded, and for y in K, $Py \geqslant Ay \geqslant Qy$. Then the positive spectrum of A lies in $[\lambda_Q, \lambda_P]$, where λ_Q and λ_P are the positive eigenvalues of Q and P which correspond to the positive eigenvectors of Q and P. If furthermore P is the Fréchet differential of A ($\lim_{|x|\to 0}|Ax-Px|/|x|=0$) and if Q is asymptotically close to A ($\lim_{|x|\to\infty}|Ax-Qx|/|x|=0$). Then the positive spectrum of A is the interval $[\lambda_Q, \lambda_P]$.

There is also developed a theory of spectral multiplicity for x_0-concave operators: A is called x_0-concave in case there is an x_0 in K such that for y in K, $y\neq 0$, one can find positive numbers μ, ν (depending on y) such that $\mu x_0 \leqslant Ay \leqslant \nu x_0$, and furthermore, if for each $y \geqslant \gamma x_0$, $\gamma>0$, for all t satisfying $0<a \leqq t \leqq b<1$, the relation $A(ty) \geqslant (1+\eta)tAy$, where $\eta>0$ depends on y, a, b. Theorem: If A is x_0-concave, completely continuous, asymptotically close to a linear operator Q and has a Fréchet differential P at 0, then the positive spectrum of A is an interval $[\alpha, \beta]$, $\beta>0$, an eigenvalue of P, and α is either 0 or a positive eigenvalue of Q. Furthermore, the positive eigenvectors of A connect 0 to ∞ and each eigenvalue in the positive spectrum of A gives rise to one and only one positive eigenvector different from 0.

B. R. Gelbaum (Minneapolis, Minn.).

Krasnosel'skiĭ, M. A., and Ladyženskiĭ, L. A. Conditions for complete continuity of P. S. Uryson's operator acting in the space L^p. Trudy Moskov. Mat. Obšč. **3**, 307–320 (1954). (Russian)

The complete continuity of the operator

$$\mathfrak{A}(\varphi)=\int_G K(x, y, \varphi(y))dy$$

considered as a transformation from L_{p_1} to L_{p_2}, $p_1, p_2>1$, is investigated. The main result is: Let $K(x, y, u)$ be continuous in u and measurable, for fixed u, in the pair (x, y). Furthermore, assume $|K(x, y, u)| \leqq R(x, y)f(y, u)$, where $|f(y, u)| \leqq a+b|u|^{p_1/p_2}$ (a, b constants, $p_1, p_2>1$) and

$$\int_G\int_G |R(x, y)|^{\max[p_2, p_2/(p_2-1)]}dxdy<\infty \quad (p_2>1).$$

(G is a bounded measurable set in n-dimensional Euclidean space, and $-\infty<u<\infty$.) Then K is completely continuous from L_{p_1} to L_{p_2}. The syllogism begins with two basic lemmas: (1) Let B be a set of finite measure in n-dimensional Euclidean space, $K(t, u)$ continuous in u for $a \leqq u \leqq b$ for almost all t in B and measurable in t, and assume for every measurable $\psi(t)$ such that $a \leqq \psi(t) \leqq b$, $t \in B$, that $\int_B |K(t, \psi(t))|dt<\infty$. Then $\int_B \sup_{a\leqq u\leqq b}|K(t, u)|dt<\infty$. (2) Under the same continuity and measurability assumptions on K and B, for each $\delta>0$, $\epsilon>0$, there is a closed set $B_0 \subset B$, such that $m(B-B_0)<\delta$, and a function $K_0(t, u)$, continuous in (t, u), such that $|K(t, u)-K_0(t, u)|<\epsilon$ for t in B_0, $a \leqq u \leqq b$. Other theorems in the train of ideas establish complete continuity for K between L_{p_1} and L_{p_2} under more restrictive conditions.

B. R. Gelbaum.

Pellegrino, Franco, e Succi, Francesco. Fondamenti della teoria dei funzionali misti complessi. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12**, 105–162 (1953).

Del Pasqua, Dario, e Pellegrino, Franco. Linee quasi analitiche dello spazio di Fantappiè e indicatrici dei funzionali misti analitici quasi lineari. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12**, 188–228 (1953).

Ces deux travaux représentent une contribution au perfectionnement topologique de la théorie de Fantappiè.

Dans le premier travail, les auteurs étudient les rapports entre les notions de "fonctionnelle mixte" et de "opérateur". Ils considèrent l'espace $\mathfrak{S}_C$ des fonctions complexes $\varphi(z)$ définies dans des ouverts D de la sphère de Riemann, $\hat{\mathbf{C}}$, avec la topologie déterminée par la définition suivante de voisinage: étant donnés $\varphi_0 \in \mathfrak{S}_C$, $\epsilon>0$, et un ensemble fermé A contenu dans le domaine de φ_0, on appele voisinage (A, ϵ) de φ_0 l'ensemble des $\varphi \in \mathfrak{S}_C$ dont les domaines contiennent A et telles que $|\varphi(z)-\varphi_0(z)|<\epsilon$ sur A. (L'espace $\mathfrak{S}_P$ des fonctions localement analytiques s'annulant à l'infini, est un sous-espace de $\mathfrak{S}_C$.) On nomme fonctionnelle mixte toute fonction complexe $F_t[\varphi(t), z]$ définie dans une partie $\mathfrak{R}$

de $\mathfrak{S}_C \times \hat{\mathbf{C}}$, vérifiant certaines conditions; l'application $\varphi \to F(\varphi, z)$ de $\text{proj}_\varphi \mathfrak{R}$ dans $\mathfrak{S}_C$ est dite l'opérateur correspondant à la fonctionnelle $F(\varphi, z)$. Les auteurs établissent beaucoup de théorèmes, dans lesquels ces deux types de transformations analytiques, considérés aux points de vue de la théorie des ensembles, de la topologie et de l'algèbre linéaire, révèlent une certaine équivalence, qui n'est pas tout à fait éclaircie.

Dans le deuxième travail, les auteurs substituent à la notion de ligne analytique de Fantappiè celle de "ligne quasi-analytique": on entend par là une application continue $\lambda \to f(\lambda, z)$ d'un ouvert D de $\hat{\mathbf{C}}$ dans $\mathfrak{S}_F$ telle que $f(\lambda, z)$ résulte analytique par rapport à λ pour chaque z. Pour qu'une application $\lambda \to f(\lambda, z)$ de D dans $\mathfrak{S}_F$ soit une ligne analytique, il faut et il suffit que $f(\lambda, z)$ soit une fonction analytique de λ, z dans un ouvert Δ de $\hat{\mathbf{C}}^2$ tel que $D = \text{proj}_\lambda \Delta$. Une fonctionnelle $F_z[\varphi(z)]$ $(\varphi \in \mathfrak{D} \subset \mathfrak{S}_F)$ est dite quasi-analytique, si $F_z[f(\lambda, z)]$ est une fonction analytique de λ pour toute ligne quasi-analytique $f(\lambda, z)$ de $\mathfrak{D}$. Dans le cas des fonctionnelles linéaires la notion de "fonctionnelle quasi-analytique" équivaut à celle de "fonctionnelle analytique". Il reste à décider la question dans le cas général.

Note du reviewer. Toutes ces recherches semblent suscitées par des critiques du reviewer, quoique les auteurs ne le déclarent pas. Il faut reconnaître que l'on a fait ici des progrès. Les résultats obtenus pourraient se condenser dans la proposition générale suivante: "Soit $\mathfrak{E}$ un espace topologique quelconque et soit $f(\mathfrak{x}, z)$ une fonction définie dans $\mathfrak{H} \subset \mathfrak{E} \times \hat{\mathbf{C}}$ et à valeurs dans $\mathbf{C}$, continue par rapport à z pour chaque $\mathfrak{x} \in \text{proj}_{\mathfrak{x}} \mathfrak{H}$, cet ensemble étant ouvert dans $\mathfrak{E}$. Alors, pour que l'application $\mathfrak{x} \to f(\mathfrak{x}, z)$ de $\text{proj}_{\mathfrak{x}} \mathfrak{H}$ dans $\mathfrak{S}_C$ soit continue, il faut et il suffit que $\mathfrak{H}$ soit ouvert dans $\mathfrak{E} \times \hat{\mathbf{C}}$ et que $f(\mathfrak{x}, z)$ soit une fonction continue de $\mathfrak{x}, z$ dans $\mathfrak{H}$". Mais on ne voit pas encore clairement l'intérêt de cette direction de recherches, après les progrès récents obtenus dans une autre direction au moyen de la théorie des espaces vectoriels topologiques. *J. Sebastião e Silva* (Lisbonne).

Del Pasqua, Dario, e Pellegrino, Franco. Sugli operatori lineari dotati di operatore associato. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12**, 35–61 (1953).

Ce travail développe les idées contenues dans une note précédente [Atti 4° Congresso Un. Mat. Ital., Taormina, 1951, v. II, Perrella, Roma, 1953, pp. 77–81; ces Rev. **15**, 133]. Mais les auteurs considèrent ici seulement de vrais espaces vectoriels, tout en évitant l'espace $\mathfrak{S}_F$ qui donne lieu à des difficultés. Ces recherches visent à une théorie, plutôt algébrique, des équations fonctionnelles linéaires.

J. Sebastião e Silva (Lisbonne).

Aruffo, Giulio, e Gallarati, Dionisio. Sulla struttura delle regioni dello spazio di Fantappiè. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12**, 1–34 (1953).

Les auteurs établissent d'abord plusieurs théorèmes concernant la réunion et l'intersection de familles de régions linéaires de l'espace $\mathfrak{S}_F$. Ensuite, ils abordent l'étude des familles de voisinages d'une fonction contenus dans un ouvert $\mathfrak{D}$ de $\mathfrak{S}_F$. Le problème de l'existence de voisinages maximaux, qui se relie à celui du prolongement analytique des fonctionnelles, est examiné en détail. Les auteurs démontrent l'existence de voisinages maximaux, avec des hypothèses assez larges, à l'aide du théorème de Zorn.

J. Sebastião e Silva (Lisbonne).

Calabi, Lorenzo. Cammini e linee analitiche in alcuni spazi funzionali. Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **12**, 62–75 (1953).

L'emploi d'un espace "universel" de fonctions holomorphes, comme celui de Fantappiè, se trouverait justifié par une théorie fructueuse du prolongement analytique des fonctionnelles. Or, la construction d'une telle théorie se relie aux concepts de connexion. Le présent travail est une contribution à l'étude de la connexion dans $\mathfrak{S}_F$ par des lignes de nature spéciale. Voici le résultat essentiel obtenu: deux points quelconques de $\mathfrak{S}_F$ peuvent toujours être unis par une ligne continue formée par cinq arcs de ligne analytique. L'auteur considère encore des espaces plus généraux que $\mathfrak{S}_F$.

J. Sebastião e Silva (Lisbonne).

Vaĭnberg, M. M. On hyperboloids and the condition for an extremum of certain functionals in Hilbert space. Uspehi Matem. Nauk (N.S.) **9**, no. 2(60), 105–112 (1954). (Russian)

Let B be a bounded operator in a real Hilbert space H, with spectral bounds $\alpha < 0$, $\beta > 0$, and let the positive spectrum of B lie in (m, β) where $m > 0$. Let E_t be the spectral resolution corresponding to B, and write $E_\beta - E_0 = P_1$, $E_0 = P_2$. We write $((x))^2 = \|P_1 x\|^2 - \|P_2 x\|^2$. Then the equation $((x)) = c$ determines a hyperboloid which is said to be generated by B. We write V for the region $((x)) = 0$.

Let $F(x)$ be a gradient of a weakly continuous functional $f(x)$, and let $(F(x), x) > 0$ if $\|x\| > 0$, while in BV let $(F(x), x) \to \infty$ as $\|x\| \to \infty$, and let the projector P_1 be finite-dimensional. Then on any hyperboloid generated by B the functional $f(Bx)$ attains its lower bound. If the condition that $(F(x), x) \to \infty$ as $\|x\| \to \infty$ is replaced by the condition that there is an increasing function $g(u)$ with $g(0) = 0$, defined in an interval $(0, a)$, such that if x is in the interval $(F(x)x, x) \geqq g(\|x\|)$, then there is an $r > 0$ such that on each hyperboloid $((x)) = c < r$ there is a closed region U relative to which $f(Bx)$ attains its lower bound at an internal point of U. *J. L. B. Cooper* (Johannesburg).

Brown, Arlen. The unitary equivalence of binormal operators. Amer. J. Math. **76**, 414–434 (1954).

The author proves the equivalence of two different definitions of a binormal operator T on a Hilbert space X, one in terms of the ring of operators generated by T and the other an explicit condition on T represented as a 2×2 operator matrix for a certain splitting of X into two orthogonal subspaces. Using these results the author easily obtains Dixmier's solution of the two-projection problem [Revue Sci. **86**, 387–399 (1948); these Rev. **10**, 546].

F. H. Brownell (Seattle, Wash.).

Cordes, Heinz Otto. Der Entwicklungssatz nach Produkten bei singulären Eigenwertproblemen partieller Differentialgleichungen, die durch Separation zerfallen. Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl. Math.-Phys.-Chem. Abt. **1954**, 51–69 (1954).

The author defines certain special operators on a direct product of two Hilbert spaces in terms of given self-adjoint operators on each factor space. He shows that these special operators are normal, have essentially only point spectra, and that the resulting complete orthonormal system of eigenvectors can be taken with each eigenvector a product of factor vectors. *F. H. Brownell* (Seattle, Wash.).

*Neumark, M. A. **Operatorenalgebren im Hilbertschen Raum.** Sowjetische Arbeiten zur Funktionalanalysis, pp. 197–274. Verlag Kultur und Fortschritt, Berlin, 1954.

Translated from Uspehi Matem. Nauk (N.S.) **4**, no. 4(32), 83–147 (1949); these Rev. **11**, 186.

*Neumark, M. A. **Involutive Algebren.** Sowjetische Arbeiten zur Funktionalanalysis, pp. 89–196. Verlag Kultur und Fortschritt, Berlin, 1954.

Translated from Uspehi Matem. Nauk (N.S.) **3**, no. 5(27), 52–145 (1948) [these Rev. **10**, 308; cf. these Rev. **12**, 111 for an English translation].

Tomita, Minoru. Representations of operator algebras. Math. J. Okayama Univ. **3**, 147–173 (1954).

This is a brief and effective attack on the problem of decomposing C^*-algebras into direct integrals of simpler pieces. The first such result is due to von Neumann [Ann. of Math. (2) **50**, 401–485 (1949); these Rev. **10**, 548], other major papers are those of Segal [Mem. Amer. Math. Soc. no. **9** (1951); these Rev. **13**, 472] and Godement [Ann. of Math. **53**, 68–124 (1951); these Rev. **12**, 421].

The chief virtue of the present method is its brevity and its complete avoidance of countability hypotheses. Basically it reduces the problem to that of the decomposition of certain states of the algebra. Begin with a C^*-algebra A over a Hilbert space H. By splitting H into a direct sum, if necessary, it can be supposed that there is a $g \in H$ such that $\{ag \mid a \in A\}$ is dense in H; Define the state p of A by $p(a) = (ag, g)$; let Z be the center of A and p_Z the state of Z determined by p. Then p is shown to be reducible if and only if p_Z is reducible, in which case $p = \int \lambda^p d\tau(\lambda)$, where for each $\lambda \in \mathfrak{Z}$, the spectrum of Z, λ^p is a uniquely determined state of A, weakly*-continuous in λ and irreducible for almost all λ.

Let E be a maximal abelian subalgebra of A' and apply the above decomposition to the state P of $A \cup E$ defined by $P(x) = (xg, g)$. Then a transformation back to X, the set of extreme points of the set of states of norm 1, yields a probability measure m on X and an expression of p as an integral over X. The other properties proved for this decomposition yield a direct integral decomposition of A and of H over X such that the terms $A(\lambda)$ are almost all irreducible; this gives Mautner's decomposition. If the theorem is applied instead to $M \cup N$, where N is a w^*-dense C^*-subalgebra of M', it yields a von Neumann decomposition of M and N into direct integrals of $M(\lambda)$ and $N(\lambda)$ where for almost all λ, $M''(\lambda)$ and $N''(\lambda)$ are factorized pairs.

M. M. Day (Urbana, Ill.).

Sunouchi, Haruo. A characterization of the maximal ideal in a factor of the case (II_∞). Kōdai Math. Sem. Rep. **1954**, 7 (1954).

Soit M un facteur de type II_∞ dans un espace hilbertien séparable $\mathfrak{H}$. L'auteur affirme qu'il existe un plus grand idéal bilatère dans M (ce qui est exact), ensemble des $A \in M$ tels que l'adhérence de $A(\mathfrak{H})$ soit de dimension finie (ce qui est faux; l'erreur se trouve à la fin: "Hence there exists the inverse $B^{-1} \in M$"). *J. Dixmier* (Paris).

Nakamura, Masahiro, and Turumaru, Takasi. On a proof of a theorem of Rosenberg. Proc. Japan Acad. **29**, 501–502 (1953).

The authors wish to withdraw this paper since it contains an error.

Goldhaber, J. K., and Wolk, E. S. Maximal ideals in rings of bounded continuous functions. Duke Math. J. **21**, 565–569 (1954).

Let C^* be the ring of all continuous bounded functions from a completely regular topological space X into a topological division ring A. (A set $B \subset A$ is bounded if for any neighborhood N of 0 there exist neighborhoods W_1 and W_2 such that $BW_1 \subset N$ and $W_2 B \subset N$. A function is bounded if its range is a bounded set.) Let A satisfy the first axiom of countability, have a continuous inverse, and have the property that if $E \subset A$ is bounded away from 0 then E^{-1} is bounded. Let S be any maximal family of countably infinite, discrete, closed subsets of X (non-compact) which is closed under union and intersection, and define $M(S) = \{f \in C^* \mid \alpha \in S$ implies $f \to 0$ on a subsequence of $\alpha\}$. Then $M(S)$ is a proper maximal ideal of C^*. The main result of the paper is that $C^*/M(S)$ is isomorphic to A if and only if A is locally compact. The authors ask the questions: 1. Is every free maximal ideal of C^* of the form $M(S)$? 2. If A is locally compact, is C^*/M isomorphic to A for every maximal ideal M? They note that an affirmative answer to 1 would also settle 2. [Ellen Correl and M. Henriksen have announced the following answers (unpublished) to these questions: 1. No. 2. Yes.] *M. Jerison* (Lafayette, Ind.).

Wermer, J. On a class of normed rings. Ark. Mat. **2**, 537–551 (1954).

The author considers certain pairs L, $\bar{L}$ of spaces of functions on the real line which generalise the prototype pair L^1, L^∞, and extends to such pairs various known results about the ring structure of L^1 and spectral synthesis in L^∞. The complete list of hypotheses is too long to include here, but for the most part L is assumed to be a Banach algebra under convolution and to be invariant under the translation operators T_τ: $f(x) \to f(x-\tau)$ (τ real), whose norms on L satisfy the condition $\|T_\tau\| = O(e^{\epsilon|\tau|})$ for every $\epsilon > 0$. J is used to denote the class of such algebras in view.

In Section 1 are discussed, for algebras L in J, several standard questions in the theory of normed algebras. Denoting by $\hat{f}$ the Fourier transform

$$\hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx,$$

these questions are: (A) When does L contain functions f for which $\hat{f}(0) \neq 0$ and $\hat{f} = 0$ outside any preassigned (small) neighbourhood of $s = 0$?; (B) are the f in L, for which $\hat{f}$ has a bounded support, dense in L?; (C) has every proper closed ideal in L at least one zero? It is shown (Theorem 1) that each of (A), (B) or (C) is ensured by suitable restrictions on the rate of growth of $\|T_\tau\|$, in particular that (A) holds in L whenever

$$\int_{-\infty}^{\infty} (1+\tau^2)^{-1} \log \|T_\tau\| d\tau < \infty.$$

In Section 2 neither the ring structure of L, nor its invariance under translation, is used, attention being switched to $\bar{L}$ and problems of weak approximation by trigonometric polynomials. Given $\varphi \in \bar{L}$, S_φ denotes the intersection of all closed sets Λ which synthesise φ in the sense that φ is the weak limit of finite linear combinations of exponentials $e^{i\lambda x}$ with $\lambda \in \Lambda$. Question: Under what conditions is S_φ non-void for every $\varphi \neq 0$ in $\bar{L}$? Sufficient conditions are given (Theorem 2) involving the rate of growth of $\|e^{-\sigma|x|}\|$ as $\sigma \downarrow 0$ (norm in $\bar{L}$ as the dual of L). It is also shown (Theorem 2′)

that any $\varphi \in \bar{L}$ is synthesisable from any neighbourhood of $S_\varphi \cup \{\infty\}$.

Section 3 returns to algebras L in J and deals with the further problem (D): How can one characterise the primary ideals in L? If I is an ideal in L, $h(I)$ denotes the set of zeros of I; and if $\varphi \in \bar{L}$, I_φ is the closed ideal in L formed of the $f \in L$ satisfying $f * \varphi = 0$. Two results are proved. Theorem 3: If (A) and (B) hold in L, then $S_\varphi = h(I_\varphi)$ for each $\varphi \in \bar{L}$. Theorem 4: If $\|T_\tau\| = O(|\tau|^k)$ ($|\tau| \to \infty$; some $k > 0$), to any closed ideal I in L admitting a unique zero p corresponds an integer $n \leqq k$ such that I consists precisely of those $f \in L$ for which $\hat{f}$, together with its first n derivatives, vanish at p.

The techniques involve results of Paley and Wiener and of Levinson about Fourier transforms, and earlier work of Beurling. *R. E. Edwards* (London).

Theory of Probability

Lukacs, Eugene, and King, Edgar P. A property of the normal distribution. Ann. Math. Statistics **25**, 389–394 (1954).

Let $X_1, \cdots, X_n$ be independent random variables with finite nth moments. The two linear forms $\sum_1^n a_\nu X_\nu$ and $\sum_1^n b_\nu X_\nu$ are independent if and only if each X with a non-zero coefficient in both forms is normally distributed and $\sum_1^n a_\nu b_\nu \operatorname{var}(X_\nu) = 0$. It is acknowledged that an equivalent result without the assumption of moments was announced by Skitovič [Doklady Akad. Nauk SSSR (N.S.) **89**, 217–219 (1953); these Rev. **14**, 1098]. *K. L. Chung.*

Deuker, E. A. Über die Verteilungsfunktionen von Vektorsummen. Z. Angew. Math. Mech. **34**, 162–174 (1954). (English, French, and Russian summaries)

Let l be the length of the sum of n independent p-dimensional vectors whose directions are uniformly distributed. For the case when the lengths $a_1, a_2, \cdots, a_n$ of the vectors are fixed, the distribution function of l is obtained as an integral whose integrand involves a product of Bessel functions. In cases where the a_k are random variables (independent of the directions) with distributions of certain types, the author obtains, among other results, extensions of the well-known reproductive properties of the normal and the Cauchy distribution. *W. Hoeffding.*

Tiago de Oliveira, J. Sur le calcul des moments de la réciproque d'une variable aléatoire positive de Bernoulli et Poisson. Anais Fac. Ci. Porto **36**, 165–168 (1952).

Let X be a random variable having a frequency function $\Pr(X=k) = \binom{n}{k}(1-q)^k q^{n-k}(1-q^n)^{-1}$. The author proposes to compute the moments of the reciprocal of X by solving numerically the differential equations they satisfy as functions of q. A similar computation can be made for positive Poisson variables. *L. LeCam* (Berkeley, Calif.).

Homma, Tsuruchiyo. On a convoluted Poisson process. Rep. Statist. Appl. Res. Union Jap. Sci. Eng. **3**, 6–12 (1953).

The author studies convolutions of Poisson processes and compound Poisson processes generalizing results of A. Rényi [Publ. Math. Debrecen **2**, 66–73 (1951); these Rev. **13**, 51]. *E. Lukacs* (Washington, D. C.).

Bellman, Richard. Limit theorems for non-commutative operations. I. Duke Math. J. **21**, 491–500 (1954).

Let A and B be (2×2) matrices such that $AB \neq BA$. Let $Z_1, Z_2, \cdots$ be independent chance matrices with the common distribution given by $P\{Z_i = A\} = P\{Z_i = B\} = \frac{1}{2}$ for all i. The author studies the asymptotic (with N) behavior of the chance matrix

$$Z_N Z_{N-1} \cdots Z_1 = \begin{Bmatrix} x_{1N} & x_{2N} \\ x_{3N} & x_{4N} \end{Bmatrix} \quad \text{(say)}.$$

He shows how to obtain exact and limiting expressions for the moments of x_{iN}. He proves that $E(\log x_{iN}) \sim N\alpha_i$ where α_i is a function of A and B; this is done under the assumption $|A| > 0$, $|B| > 0$, and the author announces that he has succeeded in eliminating this restriction. The results can be extended to more general cases and to matrices of higher order. *J. Wolfowitz* (Ithaca, N. Y.).

Wang, Shou-jen. Proof of a probability theorem related to complete convergence by the method of the characteristic function. Acad. Sinica Science Record **4**, 201–208 (1951). (Chinese summary)

Erdös [Ann. Math. Statistics **20**, 289–291 (1949); these Rev. **11**, 40] gave a necessary and sufficient condition for the convergence of $\sum_n P(|S_n| > n)$ where S_n is the nth partial sum of a sequence of independent, identically distributed random variables. Using the original method of Hsu and Abel summability, the author now gives a new proof by means of characteristic functions. It may be remarked that Abel summability was employed by Chung and Fuchs [Mem. Amer. Math. Soc. no. 6 (1951); these Rev. **12**, 722]. *K. L. Chung* (Syracuse, N. Y.).

Lampard, D. G. Generalization of the Wiener-Khintchine theorem to nonstationary processes. J. Appl. Phys. **25**, 802–803 (1954).

The relation between correlation function and power spectrum is generalized. The dual interpretation in terms of temporal and ensemble averages not being possible for nonstationary processes, the extension refers to ensemble averages. *H. Wold* (Uppsala).

Itô, Hirosi. The observation theory of the stationary random process. Proc. Japan Acad. **29**, 305–310 (1953).

Let $x(t)$ be a stationary random process representing an electromagnetic signal. The possibility of locating the energy carried by the signal simultaneously on the time and frequency axes is investigated and uncertainty relations of the Heisenberg type are obtained. [For a more detailed study see Blanc-Lapierre and Fortet, "Théorie des fonctions aléatoires", Masson, Paris, 1953; these Rev. **15**, 883; and A. Tortrat, J. Math. Pures Appl. (9) **32**, 85–128 (1953); these Rev. **15**, 217.] *L. LeCam* (Berkeley, Calif.).

Bass, Jean. Sur les fonctionnelles caractéristiques des solutions aléatoires de certaines équations aux dérivées partielles. C. R. Acad. Sci. Paris **238**, 561–563 (1954).

The author considers random solutions $u(x, t)$ of the nonlinear equation $u_t + uu_x = \mu u_{xx}$ introduced and extensively used by J. M. Burgers as a "one-dimensional model of turbulence" [Nederl. Akad. Wetensch., Proc. **53**, 247–260, 393–406 (1950); these Rev. **11**, 752; **12**, 648]; the type of random solutions under consideration is not specified; the main statement is that, in order to describe the statistical properties of these random solutions, one must introduce the "characteristic functional" defined by E. Hopf [J. Rational

Mech. Anal. 1, 87–123 (1952); these Rev. 15, 478] for random solutions of the Navier-Stokes equations.

J. Kampé de Fériet (Lille).

***Selected papers on noise and stochastic processes.** Dover Publications, Inc., New York, 1954. 337 pp. Cloth $3.50; paper $2.00.

Reproduction by photo-offset of the following six papers: Chandrasekhar, Rev. Modern Physics 15, 1–89 (1943) [these Rev. 4, 248]; Uhlenbeck and Ornstein, Physical Rev. (2) 36, 823–841 (1930); Wang and Uhlenbeck, Rev. Modern Physics 17, 323–342 (1945) [these Rev. 7, 130]; Rice, Bell System Tech. J. 23, 282–332 (1944); 24, 46–156 (1945) [these Rev. 6, 89, 233]; Kac, Amer. Math. Monthly 54, 369–391 (1947) [these Rev. 9, 46]; and Doob, Ann. of Math. (2) 43, 351–369 (1942) [these Rev. 4, 17].

Meyer, M. A., and Middleton, David. On the distributions of signals and noise after rectification and filtering. J. Appl. Phys. 25, 1037–1052 (1954).

Hoffman, William C. The joint distribution of n successive outputs of a linear detector. J. Appl. Phys. 25, 1006–1007 (1954).

Grosjean, Carl C. The exact mathematical theory of multiple scattering of particles in an infinite medium. Verh. Kon. Vlaamse Acad. Wetensch. Lett. Schone Kunst. België 13, no. 36, 156 pp. (1951). (Dutch summary)

The present paper deals with the subject indicated in the title by the method which the author has described earlier for treating the problem of random flights [Physica 19, 29–45 (1953); these Rev. 14, 772]. A typical problem considered is the following. In an infinite homogeneous medium a point source is emitting particles at a constant rate. These particles suffer collisions which result in scattering and absorption; to find the stationary distribution of the density of particles in the medium. This problem is solved in the following manner.

Let $f_n(l)dl$ $(n=1, 2, \cdots)$ represent the probability that a particle after the $(n-1)$th scattering will describe a trajectory of length between l and $l+dl$ in any direction. Further, let $\phi_n(r)dr$ represent the probability that a particle would undergo the nth scattering in a spherical shell between r and $r+dr$. Then from the geometry of the problem and the definition of ϕ_n the author deduces the recurrence relation

$$\phi_1(r)=f_1(r) \tag{1}$$

and

$$\phi_n(r)=\tfrac{1}{2}r\int_0^\infty dy\frac{f_n(y)}{y}\int_{|r-y|}^{r+y}dx\frac{\phi_{n-1}(x)}{x} \qquad (n=2, 3, \cdots). \tag{2}$$

In terms of the sine-Fourier transform $g_n(u)$ of $\phi_n(r)/r$ the foregoing recurrence relation is equivalent to

$$g_n(u)=\frac{g_{n-1}(u)}{n}\int_0^\infty f_n(y)\sin yu\frac{dy}{y} \quad (n=3, 4, \cdots).$$

Hence

$$g_n(u)=g_2(u)\prod_{k=3}^{n}\left[\frac{1}{u}\int_0^\infty f_k(y)\sin yu\frac{dy}{y}\right].$$

(The reduction of g_2 requires a separate discussion.) From this the solution for ϕ_n (after applying an inverse sine-transform) is obtained in the form

$$\phi_n(r)=\frac{2r}{\pi}\int_0^\infty\frac{du}{u^{n-1}}\sin ru\left[\prod_{k=1}^{n}\int_0^\infty\frac{dx}{x}f_k(x)\sin xu\right]. \tag{3}$$

The time-dependent version of the foregoing problem occurs when all particles are assumed to move with a velocity v and we are required to find the probability $\phi_n(r; t)drdt$ that a particle emitted by the point source at time $t=0$ would undergo the nth scattering in the spherical shell $(r, r+dr)$ and within the time interval $(t, t+dt)$. In this case relations (1) and (2) are replaced by

$$\phi_1(r; t)=f_1(r)\delta(t-r/v) \tag{4}$$

(where δ denotes Dirac's δ-function) and

$$\phi_n(r; t)=\tfrac{1}{2}r\iint_G\phi_{n-1}(x; t-y/v)f_n(y)\frac{dxdy}{xy} \qquad (n=2, 3, \cdots), \tag{5}$$

where G is the domain in the (x, y)-plane bounded by the lines joining $(0, r)$, $(r, 0)$; the 45° line through these two points and the horizontal line $y=vt$. If $\phi^*(r, S)$ represents the Laplace transform (with respect to t) of $\phi_n(r, t)$ then it follows from (5) that

$$\phi_n^*(r; s)=\frac{r}{2}\int_0^\infty\frac{dy}{y}f_n(y)e^{-sy/v}\int_{|r-y|}^{r+y}\frac{dx}{x}\phi^*_{n-1}(x; s).$$

Accordingly the solution of these equations is (cf. (3))

$$\phi_n^*(r; s)=\frac{2r}{\pi}\int_0^\infty\frac{du}{u^{n-1}}\sin ru\left[\prod_{k=1}^{n}\int_0^\infty\frac{dx}{x}f_k(x)e^{-sx/v}\sin xu\right].$$

Similar methods are used to derive the solutions for the case when there is an infinite plane source which emits particles isotropically at a constant rate and various variants of this case. Both stationary and non-stationary cases are treated.

In Part II of this paper the methods described are extended to the situation when at each collision the particles are scattered anisotropically. In this case the relevant quantities are expanded in spherical harmonics and Fourier-Bessel expansions are used. The details are too complicated to summarize. *S. Chandrasekhar* (Williams Bay, Wis.).

Grosjean, Carl C. The theory of multiple scattering of particles emitted by a non-isotropic plane δ-source. Meded. Kon. Vlaamse Acad. Wetensch. Lett. Schone Kunst. België 14, no. 13, 33 pp. (1952). (Dutch summary)

One of the problems considered in the paper reviewed above relates to the case of an infinite plane source of particles which are emitted isotropically. In this paper the treatment is generalized to the case when the emission by the plane source is anisotropic. *S. Chandrasekhar.*

Mathematical Statistics

Hiraga, Yoshihiko, Morimura, Hidenori, and Watanabe, Hisao. Tables for three-sample test. Ann. Inst. Statist. Math. Tokyo 5, 97–102 (1954).

Consider random arrangements of n elements of three kinds, for example n_1 elements of one kind, n_2 elements of a second kind and n_3 elements of a third kind with $n=n_1+n_2+n_3$, which may be regarded as three samples of

size n_1, n_2, n_3, and let r_i denote the number of runs of elements of the ith kind, and put $r=r_1+r_2+r_3$, that is, the total number of runs. According to Mood [Ann. Math. Statistics 11, 367–392 (1940); these Rev. 2, 228], the tabulation of the exact distribution of r reduces to that of the function which represents the number of different arrangements of r_1 objects of one kind, r_2 objects of a second kind, and r_3 objects of a third kind so that no two adjacent objects are of the same kind. Table 1 gives the largest integer, r_0, for which the probability $(r \leqq r_0) \leqq .05$ for $1 \leqq n_1 \leqq n_2 \leqq n_3 \leqq 10$, calculated from the exact distribution of r. Table 2 gives values of r_0 for 5% level of significance for $n_1 \leqq n_2 \leqq n_3$ from 10 to 30 inclusive and Table 3 gives the values of r_0 for all values of $n_1=n_2=n_3$ from 11 to 30 inclusive at the 1% and 5% levels of significance. (Extract from the paper.)

L. A. Aroian (Culver City, Calif.).

Harley, B. I. A note on the probability integral of the correlation coefficient. Biometrika 41, 278–280 (1954).

The probability integral of the correlation coefficient r is calculated numerically for $E(r)=0.9$ by expansion of the frequency function of $\tanh^{-1}(r)$ in a Gram-Charlier series.

P. Whittle (Wellington).

Bhatt, N. M. Sextiles and octiles with the ordinates at these quantiles of the standardized Pearson's type III distribution. J. Maharaja Sayajirao Univ. Baroda 2 (1953), no. 2, 117–124 (1954).

By use of Salvosa's tables of the type III distribution [Ann. Math. Statistics 1, 191–198, appendix 1–187 (1930)], the author has computed the values of the sextiles and octiles with their ordinates to six decimals for $\alpha_3=.1(.1)1.1$.

L. A. Aroian (Culver City, Calif.).

Hofmann, Ludwig. Über eine elementare Herleitung der Sheppardschen Korrekturen und eine prinzipielle Bemerkung über die letzteren. Statist. Vierteljschr. 6, 119–123 (1953).

This is a quite elementary derivation of Sheppard's corrections as average corrections (carried out in detail for the first three central moments) obtained by dividing the class interval into N equal sub-intervals and assigning equal probabilities to the event that any one of the N end-points is an end-point of a class interval and finally letting N become infinite. This device has been used before, perhaps first, and in a more general fashion, by J. R. Abernethy [Ann. Math. Statistics 4, 263–277 (1933)]. The author points out that the corrections are obtained for moments about the mean of the ungrouped distribution whereas moments are calculated about the mean of the grouped distribution [see H. C. Carver, ibid. 7, 154–163 (1936)] and gives a bound for the difference in the case of the second moment.

C. C. Craig (Ann Arbor, Mich.).

Petrov, V. V. On the method of least squares and its extremal properties. Uspehi Matem. Nauk (N.S.) 9, no. 1(59), 41–62 (1954). (Russian)

Expository paper.

J. Wolfowitz (Ithaca, N. Y.).

Kamat, A. R. Distribution theory of two estimates for standard deviation based on second variate differences. Biometrika 41, 1–11 (1954).

The author considers two estimates of the variance of a series of uncorrelated normal variates with quadratically varying mean. The estimates are based on squares and absolute values of second differences respectively. An incomplete distribution theory is given for the first case, and the first four moments of each statistic are calculated.

P. Whittle (Wellington).

Tikkiwal, B. D. Optimum allocation in successive sampling. J. Indian Soc. Agric. Statistics 5, 100–102 (1953).

Chance variables are observed in k groups, the variables within each group having a common expectation. Some of the chance variables in the kth group are correlated with variables in other groups. The expectation M_k of a chance variable in the kth group is to be estimated by that unbiased linear function of all the observations which has minimum variance. The proportion of variables in any group which are correlated with variables in other groups can be set by the experimenter. The author finds the proportion which minimizes the variance of the estimate of M_k.

L. Weiss (Charlottesville, Va.).

Narain, R. D. On the recurrence formula in sampling on successive occasions. J. Indian Soc. Agric. Statistics 5, 96–99 (1953).

Chance variables are observed in k groups, the variables within each group having a common expectation. Some of the chance variables in the kth group are correlated with variables in other groups. The expectation M_k of a chance variable in the kth group can be estimated by using only those variables in the group uncorrelated with variables in other groups, and also regression estimates of M_k can be constructed using correlated variables. The author constructs that linear combination of these two types of estimates which is an unbiased estimate of M_k and has minimum variance.

L. Weiss (Charlottesville, Va.).

Masuyama, Motosaburo. A note to "Rapid methods of estimating the sum of specified areas in a field of given size." Rep. Statist. Appl. Res. Union Jap. Sci. Eng. 3, 54 (1954).

See same Rep. 2, no. 4, 113–119 (1953); these Rev. 15, 332.

Sterne, Theodore E. Some remarks on confidence or fiducial limits. Biometrika 41, 275–278 (1954).

The confidence intervals customarily used for binomial probabilities are the "equal tails" confidence intervals tabled by C. J. Clopper and E. S. Pearson [Biometrika 26, 404–413 (1934)]. Confidence limits can also be selected on the basis of the probability of obtaining a number of successes as probable as or less probable than the observed number of successes in n trials. Such confidence intervals correspond to the shortest acceptance sets. Tables are given for $n=1$ to 10 and confidence coefficients .50 and .90.

L. LeCam (Berkeley, Calif.).

Box, G. E. P., and Hunter, J. S. A confidence region for the solution of a set of simultaneous equations with an application to experimental design. Biometrika 41, 190–199 (1954).

We are given the k linear equations

$$a_{i0}+\sum_j a_{ij}x_j=0 \quad (i, j=1, 2, \cdots, k),$$

where the coefficients $a_{10} \cdots a_{kk}$ follow a multivariate normal distribution with covariance matrix $\Omega\sigma^2$, known apart from the factor σ^2, of which an estimate s^2 is available, distributed as $\chi^2\sigma^2/\varphi$ with φ degrees of freedom, and independently of the errors in the a_{ij}. Then a joint confidence region for the

solutions x_i, with confidence probability $(1-\alpha)$, has a boundary defined by the values of the x_i which satisfy the equation

$$\begin{vmatrix} s^2kF_\alpha & \delta' \\ \delta & V \end{vmatrix} = 0,$$

where δ is the column vector formed by the left sides of the original equations, and the matrix V is a function of Ω, defined by the equation $V\sigma^2 = E(\delta\delta')$. Examples are presented to show how the shape of the boundary depends both on the covariance matrix and the 'conditioning' of the linear equations. The application of the result to the determination of a stationary point on a second-degree surface (as occurs in one method of experimentally determining the maximum of a function of several variables) is illustrated.

W. G. Cochran (Baltimore, Md.).

Hajós, G., and Rényi, A. Elementary proofs of some basic facts concerning order statistics. Acta Math. Acad. Sci. Hungar. **5**, 1–6 (1954). (Russian summary)

This is a continuation of a paper by Rényi [same Acta **4**, 191–231 (1953); these Rev. 15, 885]. The results are for the most part well-known. The main purpose of the paper is to give an elementary treatment which may be of methodological interest. The basic tool used is the probability integral transformation. *B. Epstein* (Detroit, Mich.).

Lancaster, H. O. A reconciliation of χ^2, considered from metrical and enumerative aspects. Sankhyā **13**, 1–10 (1953).

R. A. Fisher suggested [Economica **3**, 139–147 (1923); J. Roy. Statist. Soc. **87**, 442–450 (1924)] that there was a connection between the χ^2 goodness-of-fit test and the deviations of the sample moments from their expected values. The author follows up this idea and shows how certain tests of significance and the χ^2 goodness-of-fit test can be related. The relation is obtained via orthogonal transformations derived from the orthogonal polynomials appropriate to the particular distribution being tested. The relation between the two kinds of tests is most readily shown in the case of discrete distributions. The author gives details and examples for the binomial and multinomial cases. The last part of the paper extends the ideas to two-way classifications. The principal result in this case is a relation between the χ^2 of the contingency table and the correlation coefficient between the two variables used in the two-way classification.

B. Epstein (Detroit, Mich.).

Walter, Edward. χ^2-Test zur Prüfung der Symmetrie bezüglich Null. Mitteilungsblatt Math. Statist. **6**, 92–104 (1954).

Given $x_i = y_{i1} - y_{i2}$, the author is interested in testing the hypothesis $f_i(x_i) = f_i(-x_i)$, i.e. the symmetry of $f_i(x_i)$, the probability density, under the assumption of continuity of $F_i(x_i)$, the distribution function, $i = 1, 2, \cdots, n$. Arrange the n observations in order of increasing absolute value $x_{(1)}, x_{(2)}, \cdots, x_{(n)}$. Divide the n observations into m classes, the first class containing the observations $x_{(1)}, x_{(2)}, \cdots, x_{(k_1)}$, the second class $x_{(k_1+1)}, \cdots, x_{(k_1+k_2+1)}$, and so on, the mth class containing observations $x_{(n-k_m+1)}, x_{(n-k_m+2)}, \cdots, x_{(n)}$, where $k_i = k$, $i = 1, 2, \cdots, p$, and $k_i = k-1$, $i = p+1, p+2, \cdots, m$, $n' = n-m+1 = pk+(m-p)(k-1)$, $p \leqq m$, while the $m-1$ observations $x_{(k_1+1)}, x_{(k_1+k_2+1)}, \cdots, x_{(n-k_m)}$ are excluded from any of these classes. Let V_i represent the number of positive observations in the ith class. As a test of the hypothesis of symmetry, the author uses the statistic

$$\chi^2 = \frac{4}{k}\sum_{i=1}^{p}\left(V_i - \frac{k}{2}\right)^2 + \frac{4}{k-1}\sum_{i=p+1}^{m}\left(V_i - \frac{k-1}{2}\right)^2,$$

and proves that for $m \geqq 2$ this test is consistent and strictly unbiased from the power point of view, while for $m = 1$ this is not so. The omitted $m-1$ observations may be used with $n' = n$, but now the test is biased. For $m = 2$ the author provides tables of $\chi^2 = \chi^2_{\alpha,m,n'}$; $n' = 6(1)30, \infty$, to 2 or 3 decimals with α approximately equal to 0.05, 0.01, and 0.001, since the discontinuous nature of $\chi^2_{\alpha,m,n'}$ does not allow critical values of α exactly equal to 0.05, 0.01, and 0.001.

L. A. Aroian (Culver City, Calif.).

***Lawley, D. N. A modified method of estimation in factor analysis and some large sample results.** Uppsala Symposium on Psychological Factor Analysis, 17–19 March 1953, pp. 35–42. Ejnar Munksgaard, Copenhagen; Almqvist and Wiksell, Stockholm, 1953. 10 Danish crowns; 7.50 Swedish crowns.

In one model for factor analysis, the score of an individual on the ith test of a set of p tests is represented as

$$x_i = \sum_{r=1}^{k} l_{ir} f_r + e_i,$$

where f_r is the rth factor measurement, l_{ir} is the loading of the ith test in the rth factor, k the number of factors, and e_i a residual. Lawley assumes that the f_r are independent normal variates with means zero and unit standard deviations, while the e_i are independent normal variates with means zero and common variance σ^2 (unknown), and are independent of the f_r.

Let $l_r = \{l_{1r}, l_{2r}, \cdots, l_{pr}\}$ be the column vector of loadings in the ith common factor. Then the maximum likelihood estimate of l_r is

$$\hat{l}_r = (\hat{\lambda}_r - \hat{\sigma}^2)^{\frac{1}{2}} q_r,$$

where $\hat{\lambda}_r$ is the rth largest latent root of the sample covariance matrix A, $\hat{\sigma}^2$ (the estimate of σ^2) is the average of the $(p-k)$ smallest latent roots, and q_r is a column vector associated with $\hat{l}_r$ such that $q_r' q_s = \delta_{rs}$. Large-sample approximations to the variances and covariances of the $\hat{l}_{ir}$ are obtained, this being the main purpose of the paper.

W. G. Cochran (Baltimore, Md.).

Schäfer, Wilhelm. Das Mutungsproblem der Besetzungs-Verteilung. Mitteilungsblatt Math. Statist. **6**, 1–38 (1954).

Suppose that each of k balls falls at random into any one of n boxes. This paper considers the resulting distribution of s, the number of boxes occupied. The maximum likelihood estimate $\hat{k}$ of k is found and the confidence interval for k worked out. Besides the exact method for confidence intervals, two approximations are worked out: the normal approximation when $0.4n \leqq k \leqq 2.5n$, and use of the fiducial distribution of k with k set equal to $\hat{k}$. To facilitate computations a table is given of Stirling's numbers of the second kind $c_{k,s}$ to six significant figures for $k = 1(1)100$ and $s = 1(1)32$, where $s!c_{k,s} = \Delta^{s}0^{k}$.

S. W. Nash.

Ghurye, S. G., and Robbins, Herbert. Two-stage procedures for estimating the difference between means. Biometrika **41**, 146–152 (1954).

The difference between means of two populations is to be estimated by the difference between sample means on

the basis of n_1 and n_2 observations from the populations, subject to $a_1n_1+a_2n_2 \leqq A$. Let V_0 be the minimum variance of this estimator when the variances α_1^2 and σ_2^2 of the two populations are known (and the n_i are chosen accordingly). When the σ_i^2's are unknown, the authors suggest a two-stage procedure where m_i and n_i-m_i observations are taken from population i on the first and second stages, the quantities n_i-m_i being determined by estimates of the σ_i^2's from the first stage. If V^* is the variance of the resulting estimator, it is shown that $V^*/V_0 \to 1$ when $A \to \infty$ and certain conditions are satisfied. *J. Kiefer* (Ithaca, N. Y.).

Peterson, Raymond P. Density unbiased point estimates. Ann. Math. Statistics **25**, 398–401 (1954).

Let $x_1, \cdots, x_n$ be (not necessarily independent) random variables each distributed marginally according to a probability density $p(x, \theta, \sigma)$ where $\sigma \equiv (\theta_1, \theta_2, \cdots, \theta_m)$. Let $p_n(x_1, \cdots, x_n, \theta, \sigma)$ be the joint probability density of the x's. If f is a function of $x_1, \cdots, x_n$ such that $p(x, f(x_1, \cdots, x_n), \sigma)$ is measurable in $x_1, \cdots, x_n$, then the author calls f a "density unbiased point estimate of θ" when

$$\int p(x, f(x_1, \cdots, x_n), \sigma) p_n(x_1, \cdots, x_n, \theta, \sigma) \, dx_1 \cdots dx_n = p(x, \theta, \sigma'),$$

where perhaps $\sigma' \neq \sigma$. Among such "estimates" one is called "best" if it minimizes the distance from σ' to σ. These definitions are then applied to the estimation of a location-parameter θ in the presence of an unknown scale-parameter σ and a number of theorems are proved. *D. G. Kendall.*

Matusita, Kameo. On the estimation by the minimum distance method. Ann. Inst. Statist. Math. Tokyo **5**, 59–65 (1954).

The author proves a number of results, of which the following is typical and perhaps the most important consequence of several of the others. Let $F(\theta)$ be a distribution on k discrete points which depends upon a parameter θ and assigns probabilities $p_1, \cdots, p_k$ to each of the points. Let $S_n = (n_1, \cdots, n_k)$ describe the number of times each point occurs in a sequence of n independent observations with the distribution $F(\theta)$. The distance $\|F-S_n\|$ between F and S_n is defined by

$$\left[\sum_{i=1}^{k}\left((p_i)^{1/2}-\left(\frac{n_i}{n}\right)^{1/2}\right)^2\right]^{1/2}.$$

Let θ_n^* be that value of θ for which $\|F(\theta)-S_n\|$ is a minimum with respect to θ. Under certain conditions which, in a communication to the reviewer, the author says he has weakened, one has that Prob $\{\lim_{n\to\infty} \theta_n^* = \theta\} = 1$. *J. Wolfowitz* (Ithaca, N. Y.).

Matusita, Kameo. A remark to "On the estimation by the minimum distance method." Ann. Inst. Statist. Math., Tokyo **6**, 124 (1954).

See the paper reviewed above.

Barnard, G. A. Simplified decision functions. Biometrika **41**, 241–251 (1954).

Although the author states that his results are "indirectly related" to Wald's, they are clearly special cases of some of Wald's results when only two decisions are possible, using Wald's methods. Thus, for example, the author reproves a special case of the well-known separation theorem for finite-dimensional convex sets. He is also under the misimpression that there is something new in his statement of a familiar equation of Wald [e.g., Sequential analysis, Wiley, New York, 1947, p. 196; these Rev. **8**, 593]. The only new content in the paper is the author's view that "inference" and "decision" should be separated in a problem, but a precise statement of the objectives of a statistical problem in terms of the possible "inferences" and "decisions" is lacking. *J. Kiefer* (Ithaca, N. Y.).

Blum, Julius R. Approximation methods which converge with probability one. Ann. Math. Statistics **25**, 382–386 (1954).

The stochastic approximation methods of (A) Robbins and Monro [same Ann. **22**, 400–407 (1951); these Rev. **13**, 144; also Wolfowitz, ibid. **23**, 457–461 (1952); these Rev. **14**, 299] and of (B) Kiefer and Wolfowitz [ibid. **23**, 462–466 (1952); these Rev. **14**, 299, 1278] were shown in the cited references to converge weakly to appropriate values, under certain conditions. The author proves strong convergence of the methods of (A) and (B) under weaker conditions than those assumed in the cited references; in the case of (A), the regression function M need only satisfy $|M(x)| < c+d|x|$ and not boundedness as previously assumed. [For the special case of uniformly bounded random variables, the result for (A) was also proved by Kiefer and Wolfowitz (whose proof the author also sketches) and (under additional restrictions mentioned in the next review) by Kallianpur.] The author also considers a modification of (A) in which $M(x)$ is the median (rather than the mean) of the distribution function $H(y|x)$, and under weak assumptions proves strong convergence of an appropriate modification of the method of (A). *J. Kiefer* (Ithaca, N. Y.).

Kallianpur, Gopinath. A note on the Robbins-Monro stochastic approximation method. Ann. Math. Statistics **25**, 386–388 (1954).

Proof of the result (A) of the preceding review under the conditions of Robbins and Monro and certain additional restrictions on the sequence $\{a_n\}$ characterizing the method. Under these restrictions, estimates of b_n (expected value of squared error after n steps) are obtained (their statement is too lengthy to be given here) which yield the result. *J. Kiefer* (Ithaca, N. Y.).

Albert, G. E. On the computation of the sampling characteristics of a general class of sequential decision problems. Ann. Math. Statistics **25**, 340–356 (1954).

For sequential procedures with finitely many decisions where the "state" after n observations may be considered as a stationary Markov chain with (possibly randomized) stopping rule not depending on the number of steps taken, the author writes down the usual integral equations (IE) for probability of making a given decision, the ASN, etc., as a function of the starting point. In the case of the Wald sequential probability ratio test between two simple Koopman-Darmois hypotheses, consideration of the IE leads to a sketch of a proof that the OC function is monotone in the parameter. The main novelty of the paper lies in the obtaining of upper and lower bounds on the solution of the IE which improve slightly on the Wald bounds for the OC function, as demonstrated in a normal example. *J. Kiefer* (Ithaca, N. Y.).

Grenander, Ulf. On the estimation of regression coefficients in the case of an autocorrelated disturbance. Ann. Math. Statistics **25**, 252–272 (1954).

The author considers a seemingly ordinary problem and derives a series of remarkable and significant results. He

examines the estimation of the coefficients $c^{(n)}$ of the regression

$$x_\nu = \sum_{n=1}^{s} c^{(n)} \varphi_\nu^{(n)} + y_\nu,$$

where the independent variables φ_ν are given functions of ν and the residual y_ν is a stationary variate, in general uncorrelated. Expressions are derived for the efficiency of the least square (L.S.) estimates relative to the best linear unbiased (B.L.U.) estimates. Conditions for the efficiency of the L.S. estimates then follow. (The L.S. estimates are defined as those obtained by minimising $\sum y_\nu^2$, the B.L.U. by minimising $\sum \eta_\nu^2$, where the η's are a set of uncorrelated variates obtained by a linear transformation of the y's with unit Jacobian.)

The φ_ν's will not in general constitute stationary sequences. However, the author shows that if $\Phi(N) = \sum_0^N \varphi_\nu^2$ tends to infinity with N while $\Phi(N+h)/\Phi(N)$ tends to unity for any finite h then the correlation coefficients of the infinite sequence may be defined satisfactorily together with a spectral function $\varphi(\lambda)$. (To call sequences obeying these conditions stationary, as the author does, is perhaps an overstrain of the term, since this would bring polynomials, for instance, into the fold of stationary variables.)

The author now shows that the asymptotic efficiency of the L.S. estimate in the case of one independent variable is $[\int f(\lambda)^{-1} d\varphi(\lambda) \int f(\lambda) d\varphi(\lambda)]^{-1}$ where $f(\lambda)$ is the spectral density of the y process, assumed positive and continuous. [The device of approximating a general type of process by an autoregression of increasing order, claimed by the author as new, has been used before, though hardly so rigorously. See Whittle, Biometrika 39, 309–318 (1952); J. Roy. Statist. Soc. Ser. B. 15, 125–139 (1953); these Rev. 14, 666; 15, 143.] The efficiency will thus be unity for all f if and only if $\varphi(\lambda)$ has only one point of increase. The generalisation of these statements to the case of several variables is by no means immediate, although the results are simple.

The author considers now analytic functions of the type

$$\varphi_\nu = \sum_{j=0}^{p} \nu^{p-j} \int_{-\pi}^{\pi} e^{i\nu\lambda} dF_j(\lambda)$$

and develops simple criteria for the efficiency of the L.S. estimates in terms of the "stressed frequencies" of the different $\varphi_\nu^{(n)}$ (the frequencies for which $F_0(\lambda)$ has a saltus). Corollaries of his general conclusions are that the L.S. estimates are always asymptotically efficient for polynomial or trigonometric regressions. *P. Whittle.*

Grenander, Ulf, and Rosenblatt, Murray. Regression analysis of time series with stationary residuals. Proc. Nat. Acad. Sci. U. S. A. 40, 812–816 (1954).

With reference to the article of the preceding review, the authors present a more explicit and elegant treatment of the case with several regression variables. *H. Wold.*

Jenkins, G. M. An angular transformation for the serial correlation coefficient. Biometrika 41, 261–265 (1954).

The author proposes to stabilise the variance of the first autocorrelation coefficient r (of a sample series generated by a first-order autoregression with $E(r) = \rho$) by the use of $\sin^{-1}(r)$. This variate is shown to be asymptotically normal, more stable than $\tanh^{-1}(r)$, and useful for test purposes when $|\rho| < 0.9$. *P. Whittle* (Wellington).

Robin, Louis. Fonction de corrélation propre et spectre de la densité de puissance du bruit thermique écrêté. Filtrage de signaux périodiques, simples, dans ce bruit. Ann. Télécommun. 7, 375–387 (1952).

The theory of noise through nonlinear devices has been studied by many investigators, in particular, Rice [Bell System Tech. J. 23, 282–332 (1944); 24, 46–156 (1945); these Rev. 6, 89, 233] and Middleton [Quart. Appl. Math. 5, 445–498 (1948); these Rev. 9, 362]. The present paper aims to cast certain results in forms easily amenable to computation. The autocorrelation function of random noise of bandwidth f_0 is given by $\phi(\tau) = \phi(0)(\sin \omega_0 \tau)/\omega_0 \tau$, where $\omega_0 = 2\pi f_0$ and $\phi(0)$ is the average noise power. $\phi(\tau)$ is normalized by taking $\phi(\tau) = (\sin \omega_0 \tau)/\omega_0 \tau = r$, and $\phi_l(\tau)$ denotes $\phi(\tau)$ when modified by the limiter $(-l, l)$. The determination of $\phi_l(\tau)$ is first carried out by a statistical method. Secondly, $\phi_l(\tau)$ is studied by the characteristic function method, resulting in the power series:

$$\phi_l(\tau) = r\Phi^2_{-1}(l) + \sum_{n=1}^{\infty} \frac{r^{2n+1}}{(2n+1)!} \Phi^2_{2n-1}(l),$$

where

$$\Phi_{-1}(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^x \exp -\tfrac{1}{2}\xi^2 \, d\xi, \quad \Phi_p(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d^p}{dx^p} \exp -\tfrac{1}{2}x^2.$$

By means of a formula for the product of Hermite polynomials due to Mehler [Erdélyi, Math. Z. 44, 201–211 (1938)] the above series can be written as a very simple definite integral. The result is applied to compute the spectrum density of power, and finally applied to the case of filtering a periodic signal and noise when passed through a nonlinear apparatus. *S. Ikehara* (Tokyo).

de Vries, M. Statistical methods in cryptanalysis. Math. Centrum Amsterdam. Rapport ZW 1953-014, 15 pp. (1953). (Dutch)

This paper is an introduction to statistical analysis of ciphers. The author treats enciphering as a succession of transformations, each taking letters of the plain text into cipher letters. If the transformations are all the same type, differing in a parameter, then the succession of parameters is called the "key". One particularly important type of transformation is the monoalphabetic substitution, by which a plain letter is taken into the same cipher each time it appears.

The concept of homogeneous text is introduced, in which all letters are equally frequent. A basic technique of analysis is to assume the text homogeneous and then search for phenomena which are extraordinary by this hypothesis. The author introduces several statistics to be examined. One, the K-test, is a measure of how far the frequencies depart from homogeneous. Another, the ϕ-test, is a test for monoalphabeticity. Two tests are given for determining whether two samples resemble each other. A test is given to determine whether two cryptograms have been prepared by the same succession of transformations. Then an example is given and broken using statistical methods almost exclusively, in which the letters are treated as residues mod 26 and the key is added to the plain text. The key is used over and over again, or one could say it is cyclic. The solution seems highly plausible.

The distributions of the various statistics are not discussed.

H. Campaigne (Washington, D. C.).

Gabor, Denis, and Gabor, André. An essay on the mathematical theory of freedom. J. Roy. Statist. Soc. Ser. A. 117, 31–60; discussion, 60–72 (1954).

This is a novel presentation of a statistical approach to freedom which was inspired by information theory. It is statistical in the sense that it relates to large populations and not to individuals. Freedom is considered to be made up of constituent factors, diversity and independence, and measures of these are proposed. Let the vector $i=(i_1, i_2, \cdots)$, consisting of components which are an individual's simple choices, denote a composite choice, and let the vector $j=(j_1, j_2, \cdots)$ consisting of components which describe an individual denote a datum. Further, let $p(i, j)$ denote the fraction of the population who have made the choice i in condition j. Call the fraction $p(i)=\sum_j p(i, j)$ a preference and set $P(j)=\sum_i p(i, j)$. Then it is shown that statistical freedom, $f_{i,j}$, of an individual, j, regarding choice i given by $f_{i,j}=D_i(1-e^{2}{}_{i,j})$, where the diversity factor is

$$D_i=-(\sum_i p_i/\log p_i)^{-1}$$

and where the coefficient of dependence is

$$\epsilon_{i,j}=-\sum_i\sum_j \frac{p(i, j)}{\log p(i)}\log\frac{p(i, j)}{p(i)[p(j)]}$$

satisfies certain reasonable postulates. Later the condition that the simple choices be discrete is removed. Examples dealing with political elections and personal choice of profession are considered numerically. The paper is followed by a lively discussion by Barnard, Champernowne, I. J. Good, M. G. Kendall, Bartlett, Vajda and MacKay in which the discussants praise the authors' courage in undertaking their study, but raise questions regarding some basic assumptions. *R. A. Leibler* (Washington, D. C.).

Mathematical Economics

Mills, W. H. The four person game—edge of the cube. Ann. of Math. (2) 59, 367–378 (1954).

The author finds all solutions in the sense of von Neumann and Morganstern of the four-person zero-sum game whose characteristic function satisfies $v(\{1, 3\})=v(\{1, 4\})=2$ and $v(\{1, 2\})=t$ where $-2\leqq t\leqq 2$. The results are too technical for a detailed statement to be given here. The solutions break up into four essential types. In the first, referred to as the trivial case, the solutions are the union of certain polyhedral sets depending on t in a rather complicated way. In the other three cases the solutions involve a certain monotone curve C in addition to certain polyhedral sets, generally segments. It is notable that the game has no finite solutions for $-2<t<2$. *D. Gale* (Providence, R. I.).

Glicksberg, I. A derivative test for finite solutions of games. Proc. Amer. Math. Soc. 4, 895–897 (1953).

Let Γ be a game on the unit square with a continuous pay-off function M such that $\partial^n M(x, y)/\partial y^n$ is continuous and of one sign for all (x, y). Theorem: There exist optimal mixed strategies for the two players using not more than n pure strategies for the maximizing player and not more than $n/2$ pure strategies for the minimizing player (counting, in this case, 0 and 1 as half strategies). This generalizes a theorem that was proved by Karlin for $n\leqq 4$.

D. Gale (Providence, R. I.).

Woodbury, Max A. Linear-convex games. J. Soc. Indust. Appl. Math. 1, 137–142 (1953).

Let Γ be a two-person zero-sum game in which the pure strategies for the first player consist of all vectors $x=(x_1, \cdots, x_m)$ where $x_i\geqq 0$ and $\sum x_i=1$, and those for the second player consist of all vectors $y=(y_1, \cdots, y_n)$ where $y_i\geqq 0$ and $\sum y_i=1$. Let the pay-off function ϕ be given by $\phi(x, y)=\sum_i x_i f(\sum_j b_{ij}y_j)$, where f is non-decreasing and convex. Let x^0 and y^0 be optimal mixed strategies for the ordinary finite game with pay-off, $\psi(x, y)=\sum_{i,j} x_i b_{ij} y_j$. It is shown that x_0 and y_0 are also optimal strategies for Γ. The result is applied in the special case where f is the exponential function to a problem of choosing weapons for a battle. The problem consists in determining what proportion of each player's financial resources should be allocated to weapons of varying effectiveness and cost in order to achieve an optimal strategy against the enemy.

D. Gale (Providence, R. I.).

Luce, R. Duncan. A definition of stability for n-person games. Ann. of Math. (2) 59, 357–366 (1954).

The purpose of the definition is to measure the ease with which disruptive coalitions can form in an n-person game. Let $v(S)$ denote the amount which the coalition S can win by optimal play against its complement $I-S$, let x be a function on I satisfying $\sum_I x(i)=v(I)$, $x(i)\geqq v(\{i\})$, all $i\in I$, and let τ be a proper partition of I. Then (x, τ) is defined to be k-stable if (i) $\sum_T x(i)\geqq v(T)$ for every T differing from an element of τ by at most k elements and (ii) $x(i)=v(\{i\})$ only if $\{i\}\in\tau$. The game is said to be k-stable if there exists such a k-stable pair. The author establishes a number of results, for example: a simple game ($v(S)=0$ or 1, all S) is k-stable if and only if the intersection of all $(k+1)$-element coalitions S with $v(S)=1$ is nonempty, and in particular is 1-stable unless it is strategically equivalent to the 3-person majority game with $n-3$ dummy players added. *L. S. Shapley* (Santa Monica, Calif.).

Burger, E. Spieltheoretische Behandlung eines Reklameproblems. (Variante eines Spielmodells von Gillman.) Mitteilungsblatt Math. Statist. 6, 39–52 (1954).

The author obtains explicit solutions for a zero-sum two-person game described in terms of an advertising competition. The model is equivalent to the silent duel in which each duellist is armed with just one bullet. Games of this sort have previously been analyzed by Blackwell and Girshick [E. W. Paxson, Econometrica 17, 72–73 (1949)] (however, the payoff function in the particular game described there is slightly different from that under review). [The reference in the title is to J. Amer. Statist. Assoc. 45, 541–545 (1950), in which the model is the analog of the single bullet vs. machine gun duel.] *L. Gillman* (Lafayette, Ind.).

Fels, Eberhard. Einige Bemerkungen zu Burgers Variante eines Spielmodells von Gillman. Mitteilungsblatt Math. Statist. 6, 53–55 (1954).

Remarks concerning the realism of the game model of the preceding review. Non-mathematical. *L. Gillman.*

Bellman, Richard. Some applications of the theory of dynamic programming—a review. J. Operations Res. Soc. Amer. 2, 275–288 (1954).

McGarvey, David C. A theorem on the construction of voting paradoxes. Econometrica 21, 608–610 (1953).

A preference pattern on a set S of n elements consists of two relations, P and I on S (xPy is read "x is preferred to y", xIy is read "x and y are indifferent"), where I is reflexive (xIx for all $x \in S$) and symmetric (xIy implies yIx for all $x, y \in S$) and P is anti-reflexive and anti-symmetric. If both P and I are also transitive, the preference pattern is called an ordering, and if the pattern contains no indifferences it is called a strong ordering. In an obvious way the method of majority decision makes correspond to any set of individual preference patterns on S a community preference pattern on S. It is well known that the community pattern obtained in this way need not be an ordering even when all individual patterns are. The author generalizes this by showing that any preassigned preference pattern can be obtained by the method of majority decision from individual preference patterns which are strong orderings.

D. Gale (Providence, R. I.).

TOPOLOGY

Turán, P. On the theory of graphs. Colloquium Math. 3, 19–30 (1954).

This paper is mainly expository. It is concerned with criteria for the existence in a given graph of a subgraph which is a complete k-graph. The author has determined the structure of those graphs of n vertices which have no complete k-graph as a subgraph and which have the maximum number of edges consistent with this condition. He naturally imposes the condition that no two vertices may be joined by more than one edge. This paper contains the first proof in English of his result. *W. T. Tutte* (Toronto, Ont.).

Capel, C. E. Inverse limit spaces. Duke Math. J. 21, 233–245 (1954).

The first part of this paper is concerned with a proof of the fact that the continuity axiom is equivalent to the extension and reduction theorems. The continuity axiom asserts that for compact Hausdorff spaces the cohomology groups of an inverse limit are the direct limits of the corresponding groups of the factor spaces. The extension theorem asserts that for a closed subset of a compact Hausdorff space any cohomology class on the subset is obtained by restriction from a class on a compact neighborhood of the subset, and the reduction theorem asserts that if a cohomology class on a compact space restricts to zero on a closed subset it restricts to zero on some compact neighborhood of the subset. Hence, the extension and reduction theorems together are equivalent to asserting that the continuity axiom holds for a closed subset of a compact Hausdorff space regarded as the inverse limit of its closed neighborhoods. The passage from this special case of the continuity axiom to the general case is accomplished in the product space of the factors where the inverse limit is a suitable intersection of closed subsets.

The second part of the paper studies limit spaces in which the maps connecting the factors are monotone (i.e. they are onto and the inverse image of every point is a continuum). It is shown that in this case local connectedness and semi-local connectedness are preserved in passage to inverse limits (for compact spaces), and that if the indexing set of the limit system is countable, the inverse limit of arcs or simple closed curves is again an arc or simple closed curve.

E. H. Spanier (Chicago, Ill.).

Yang, Chung-Tao. On paracompact spaces. Proc. Amer. Math. Soc. 5, 185–189 (1954).

The author first shows that, for paracompact spaces, normality is equivalent to local normality and to each of the following: (T_3') If two points have disjoint closures, they have disjoint neighbourhoods; (T_3'') For each point x and neighbourhood U of x there is a neighbourhood V of x such that $\bar{V} \subset U$. Pointwise paracompactness does not suffice for these equivalences, but implies that the closure of each point is compact. The effect of identifying points whose closures meet is then considered, and the following theorem is deduced. A space X is pointwise paracompact and T_3' if and only if it can be retracted onto a regular Hausdorff pointwise paracompact subset A which meets every non-null closed subset of X; A is then unique, to within a homeomorphism, and the corresponding retraction is uniquely determined by A. Further, X is normal, locally normal, or paracompact, if and only if A is. *A. H. Stone.*

Ball, B. J. Countable paracompactness in linearly ordered spaces. Proc. Amer. Math. Soc. 5, 190–192 (1954).

Let X be a linearly ordered space, the open intervals of X forming a base for the open sets. The author's proof shows that every open covering W of X, for which each point belongs to at most countably many sets of W, has a locally finite refinement. In particular, X is countably paracompact.

A. H. Stone (Manchester).

Dieudonné, Jean. Sur les sous-espaces linéairement compacts. Bol. Soc. Mat. São Paulo 6 (1951), 53–60 (1952).

(For definitions and previous related results see pp. 74–83 of Lefschetz's "Algebraic topology" [Amer. Math. Soc. Colloq. Publ., v. 27, New York, 1942; these Rev. 4, 84] and the author's papers in Amer. J. Math. 73, 13–19 (1951) and J. Reine Angew. Math. 188, 162–166 (1950) [these Rev. 12, 476, 715].) The paper contains numerous results on linearly compact vector spaces, displaying their similarity to finite-dimensional vector spaces. Some of these are as follows (where E is always a linearly topologized left vector space over a discrete sfield): (3) Every linearly compact subspace of E admits a topologically supplementary subspace. (4) If a closed subspace N of E admits a linearly compact supplementary subspace M, then M is topologically supplementary to N. (6) If M is a linearly compact and N a closed subspace of E, then $M+N$ is closed. (8) If F is a linearly compact vector space and u is a linear map of E into F whose graph is closed in $E \times F$, then u is continuous. Certain of these results (e.g., (3)) are extended to locally linearly compact spaces, while others (e.g., (4)) are shown not to extend. The paper closes with an example of a linearly topologized space E and a closed subspace M of E such that E/M is linearly compact even though M admits no topologically supplementary subspace. *V. L. Klee* (Seattle, Wash.).

Ellis, David. A set-theoretic description of normal topologies. Amer. Math. Monthly 61, 405–407 (1954).

Let X be a normal space and let U be the set of Urysohn functions on X, that is, continuous maps of X onto $[0, 1]$: for each x let $h(x)$ be the set of those u in U which vanish on some neighborhood of x. Then a net x_n, $n \in N$, N a

directed system, converges to a point x of X if and only if $\bigcup_\alpha \bigcap_{\alpha>\alpha} k(x_\alpha) \supset k(x)$. This characterizes the topological structure in X in terms of the set-theoretical properties of U.

M. M. Day (Urbana, Ill.).

Dubikajtis, L. On the separability of topological spaces. A supplement to a paper of R. Sikorski. Colloquium Math. 3, 31–32 (1954).

The author points out that one possibility was overlooked in Sikorski's enumeration of all the logical connections between six varieties of separability [Colloquium Math. 1, 279–284 (1948); these Rev. 11, 44]. The gap, once noticed, is easily filled. *A. H. Stone* (Manchester).

Morita, Kiiti. On spaces having the weak topology with respect to closed coverings. Proc. Japan Acad. 29, 537–543 (1953).

A topological space X has the weak topology with respect to a covering $\{A_\alpha\}$ by closed sets if the union of any subcollection $\{A_\beta\}$ of the $\{A_\alpha\}$ is closed in X, and if any set whose intersection with each A_β is open in A_β is open in $\bigcup_\beta A_\beta$. In such spaces, it is shown that if each A_α is metrizable as a subspace, then any subset of X is paracompact and perfectly normal. *J. Dugundji.*

Michael, Ernest. Local properties of topological spaces. Duke Math. J. 21, 163–171 (1954).

A property $\mathfrak{P}$ of topological spaces is "G-hereditary" ("F-hereditary") for a space X if, for every subspace Y of X, we have: (a) if Y has $\mathfrak{P}$, so has every relatively open (closed) subset of Y; (b) if Y is the union of two open (closed) subsets of X which have $\mathfrak{P}$ (and whose frontiers relative to Y are disjoint), then Y has $\mathfrak{P}$; (c) if Y is open (closed) in X and is the union of a disjoint collection of open-closed sets relative to Y which have $\mathfrak{P}$, then Y has $\mathfrak{P}$. The space X has $\mathfrak{P}$ "locally" if every $x \in X$ has a neighbourhood (not necessarily open) which has $\mathfrak{P}$; X has $\mathfrak{P}$ "uniformly locally" if there is an entourage V in some uniform structure on X for which $V(x)$ has $\mathfrak{P}$ for all $x \in X$. Theorem: If $\mathfrak{P}$ is either G-hereditary or F-hereditary for a space X, and if either (1) X is paracompact and has $\mathfrak{P}$ locally, or (2) X has $\mathfrak{P}$ uniformly locally, then X has $\mathfrak{P}$. For G-hereditary properties, this follows from the result that if $\mathfrak{U}$ is a normal open covering of X, then X is a finite union of discrete unions of finite unions of discrete unions of open subsets of the various sets of $\mathfrak{U}$. The proof for F-hereditary properties depends on a similar, but more complicated, result. The author shows that his theorem includes results of Hanner [Ark. Mat. 2, 315–360 (1952); these Rev. 14, 396] and Montgomery [Fund. Math. 25, 527–533 (1935)], since the properties of being a neighbourhood extensor for a given collectionwise normal space, and of being a G_δ (or Borel set of higher class) in a metric space, are G-hereditary. Similarly the properties of normality and paracompactness are F-hereditary. *A. H. Stone* (Manchester).

Isbell, J. R. More on the continuity of the real roots of an algebraic equation. Proc. Amer. Math. Soc. 5, 439 (1954).

The author shows that a theorem stated, but not proved, by Hewitt for completely regular spaces and proved by Henriksen and the author for normal spaces [same Proc. 4, 431–434 (1953); these Rev. 14, 1107] is true as originally stated.

Curtis, M. L. A note on monotone deformation-free mappings. Proc. Amer. Math. Soc. 5, 437–438 (1954).

A deformation-free mapping of a closed subset M of a space S into S is a map $h: M \times I \to S$ such that $h(x, 0) = x$ and $h(x, t)$ non-$\in M$ for $0 < t \leqq 1$. The author has previously proved [Ann. of Math. (2) 57, 231–247 (1953); these Rev. 14, 1002] that if M is a locally simply connected continuum separating the n-sphere S^n, if A is a component of $S^n - M$, and if M is deformation-free into $\bar{A}$ with a monotone open map h, then A is uniformly locally simply connected. It is proved here that the condition that h be open may be omitted. *E. E. Floyd* (Charlottesville, Va.).

Martin, Abram V. Decompositions and quasi-compact mappings. Duke Math. J. 21, 463–469 (1954).

The author's principal results provide sufficient conditions that the image of a separable metric space under a closed or quasi-compact mapping be separable and metric. It is shown that if the Hausdorff space X is the image under a quasi-compact, monotone mapping of a locally compact, separable, metric space then X is locally compact, separable and metric. If the metric space X is the image under a quasi-compact mapping of the separable metric space M, then X is separable and, if M is locally compact, X is also. If the topological space X is the image under a closed mapping of a separable metric space and if for each $x \in X$ the inverse set of x is compact, then X is separable and metric. In the last theorem quoted, X is assumed to contain a family of "open" sets which are closed under union and intersection but no separation axiom is required. Examples are given to show that the theorems do not hold upon omitting any one of the hypotheses. *W. R. Utz* (Columbia, Mo.).

Kasriel, Robert H. k-fold irreducible decomposition of a space relative to a mapping. Proc. Amer. Math. Soc. 5, 440–446 (1954).

Let f be a map of A onto B, A and B compact metric, and let $D_n = \{x \mid f^{-1}f(x) = n\}$, with n a positive integer. The author is concerned with conditions under which $D_n^* = A$, where * means closure. Denote by K_n the assertion: A is the union of n non-void sets A_i all satisfying (i) $A_i^{*\circ} = A_i$ ($^\circ$ means interior), (ii) $A_i^\circ \cap A_j^\circ$ is void if $i \neq j$, and (iii) $f(A_i) = B$ and no closed proper subset of A_i has this property. The main result can be formulated in this way: K_n implies $D_n^* = A$, and if f is open then $D_n^* = A$ implies K_n. With the aid of factorization theorems the author strengthens his result to the case where f is quasi-interior or quasi-monotone. Special choices of A are considered. The earlier proofs are novel and interesting.

A. D. Wallace (New Orleans, La.).

Anderson, R. D. Continuous collections of continuous curves. Duke Math. J. 21, 363–367 (1954).

The author has already constructed [same J. 19, 359–366 (1952); these Rev. 14, 71] a monotone interior mapping f of a one-dimensional continuum onto the Hilbert cube H. Here he shows that, by a modification of the construction, one can arrange that the continua $f^{-1}(y)$ ($y \in H$) are, moreover, locally connected (and non-degenerate). The collection of sets $f^{-1}(y)$ is necessarily continuous; whether it can be made equicontinuous is an open question. *A. H. Stone.*

Ward, A. J. A generalization of the Fréchet distance of two curves. Proc. Nat. Acad. Sci. U. S. A. 40, 598–602 (1954).

Let E be a topological space. A generalized curve representation is a continuous map p of the unit interval I:

$0 \leqq x \leqq 1$, into E such that p is not constant on any non-degenerate interval. Two representations p and p^* are equivalent if there is a homeomorphism v of I onto I such that $p \circ v = p^*$. A generalized curve h is a maximal set of equivalent representations. Morse [Topological methods in the theory of functions of a complex variable, Princeton, 1947, pp. 99–117; these Rev. 9, 20] has shown that if U is a metric space, the Fréchet distance between two (generalized) curves vanishes if and only if the curves coincide. In this paper it is merely assumed that E is a separated uniform space with uniformity U defined by a family of écarts (quasi-distances). A uniformity $\bar{U}$ is defined in a classic fashion on the set H of generalized curves on E and it is shown how $\bar{U}$ can be defined by a suitable family of derived écarts on $H \times H$. It is shown that there exists a uniform structure $\bar{U}_2$ on H, defined by a subfamily of the family of écarts which define $\bar{U}$, which is topologically equivalent to $\bar{U}$ and that $\bar{U}_2$ (and hence $\bar{U}$), is separated. An example shows that $\bar{U}$ and $\bar{U}_2$ are not necessarily the same. *G. A. Hedlund.*

Browder, Felix E. Covering spaces, fibre spaces, and local homeomorphisms. Duke Math. J. 21, 329–336 (1954).

Let X and Y be topological spaces and $f: X \to Y$ a continuous mapping. Then (X, Y, f) is called a triple. It is called a covering triple if X is a covering space over Y with f as covering mapping. The purpose of the paper is to characterize covering triples from among triples satisfying weaker structural conditions. First, it is shown under some local connectedness conditions on X and Y, that covering spaces can be distinguished among the fibre spaces of various types by the pathwise total disconnectedness of the fibres. This overlaps partially with the results of J. S. Griffin [same J. 20, 621–628 (1953); these Rev. 15, 337] and P. S. Mostert [21, 67–74 (1954); these Rev. 15, 890].

The later part of the paper is concerned with the triples (X, Y, f), where $f: X \to Y$ is a local homeomorphism. Local homeomorphisms might not be covering mappings as shown by known simple examples. It is established that, with some local connectedness conditions on Y, a local homeomorphism $f: X \to Y$ is a covering mapping provided that there exists for each point y of Y a neighborhood V such that f is a closed mapping of each component of $f^{-1}(V)$ into V. As a special case, it follows that, under the same local-connectedness conditions on Y, every local homeomorphism of X into Y which is a closed mapping is also a covering mapping.

S. T. Hu (New Orleans, La.).

Sitnikov, K. An example of a two-dimensional set in a three-dimensional Euclidean space which does not separate any region of this space. Doklady Akad. Nauk SSSR (N.S.) 94, 1007–1010 (1954). (Russian)

The author gives another contribution to the study of non-closed sets in Euclidean 3-space R^3. It is shown that there exists a two-dimensional set A in R^3 such that if U is any region in R^3, then any two points of $U-A$ may be joined by a continuum in $U-A$. Hence the two-dimensional subsets of R^3 do not coincide with those subsets which locally separate R^3 at some point. *E. E. Floyd.*

Borovikov, V. A. Construction of a zero-dimensional compactum of metric order n. Mat. Sbornik N.S. 34(76), 279–288 (1954). (Russian)

Suppose that M is a finite subset of a compact metric space Φ. The order $\sigma_a M$ of a point $a \in \Phi$ relative to M is the number of points $b \in M$ with $\rho(a, b) = \rho(a, M)$; the least upper bound of the numbers $\sigma_a M$ is the order of Φ relative to M. The least positive integer k such that for $\epsilon > 0$ there is a set M, ϵ-dense in Φ, relative to which the order of Φ is k, is called the metric order $\omega\Phi$ of Φ. Always $\omega\Phi \geqq \dim \Phi + 1$; if $\omega\Phi = \dim \Phi + 1$, Φ is called metrically correct. Sitnikov [Doklady Akad. Nauk SSSR (N.S.) 67, 229–232 (1949); these Rev. 11, 121] proved that every polyhedron is metrically correct. The author proves that not all simple arcs are metrically correct. It is shown that there exists in n-space a compact 0-dimensional set Φ of metric order n, and that every arc containing Φ is also of order n.

E. E. Floyd (Charlottesville, Va.).

Fort, M. K., Jr. A theorem about topological n-cells. Proc. Amer. Math. Soc. 5, 456–459 (1954).

The following theorem is proved: Let G be a non-empty, open, connected subset of Euclidean n-space R^n. Then G is the union of a non-decreasing sequence $E_1, E_2, \cdots$ of closed topological n-cells. From this the author deduces that there is a subset H of G such that $\dim H < n$, such that $G-H$ is open and connected, and such that all homotopy groups of $G-H$ vanish. And, if $n = 3$, H can be chosen so that $G-H$ is the image of R^3 under a piecewise linear homeomorphism.

E. E. Moise (Ann Arbor, Mich.).

Myškis, A. D. Generalizations of the theorem on a fixed point of a dynamical system inside of a closed trajectory. Mat. Sbornik N.S. 34(76), 525–540 (1954). (Russian)

Several sufficiency conditions are given for the existence of a fixed point under a multiple-valued map of a polyhedron P with nonvanishing Euler characteristic into itself and under a one-parameter semigroup of such maps. The conditions are rather involved. Some conditions include a restriction on the dimension of P ($\dim P = 1$ or $\dim P = 2$) and another condition includes a restriction on the behavior of the map along the boundary of P. It is also observed that every one-parameter semigroup of single-valued maps of P into P has a fixed point. Applications to differential equations are noted. *W. H. Gottschalk* (Philadelphia, Pa.).

Montgomery, Deane, and Zippin, Leo. Examples of transformation groups. Proc. Amer. Math. Soc. 5, 460–465 (1954).

Brouwer showed [Math. Ann. 80, 39–41 (1919)] that a periodic homeomorphism of R^3 or S^3 is topologically a rotation [see also Kerékjártó, Vorlesungen über Topologie, Bd I, Springer, Berlin, 1923; and Eilenberg, Fund. Math. 22, 28–41 (1934)]. The authors showed [Trans. Amer. Math. Soc. 40, 24–36 (1936)] that the circle group C acting on R^4 is topologically the same as rotation about a line. The authors now exhibit an action of C on R^4 for which C is not equivalent to a subgroup of the orthogonal group of R^4. The example, which has other interesting properties, takes its start in a construction given by R. H. Bing [Ann. of Math. (2) 56, 354–362 (1952); these Rev. 14, 192].

A. D. Wallace (New Orleans, La.).

Robinson, J. E. Continuity of transformation groups in topological spaces. Duke Math. J. 21, 337–348 (1954).

A T_1 space X which is homogeneous under a group G of homeomorphisms of X is said to satisfy condition A if for any open set O in X and point $p \in O$ there exists a non-empty open set U in X such that if $T \in G$ and $T(U)$ contains p, then $T(U) \subset O$. It is shown that if X is a T_1 space homogeneous under an Abelian group G of homeomorphisms of X and such that condition A is satisfied, then a group

operation may be defined in X in such a way that X becomes an Abelian topological group and G the group of translations of X. Conversely, if X is a topological group, then X satisfies condition A and is strictly homogeneous under the group of left translations of X. Examples are given in which X is metric and homogeneous under an Abelian group yet fails to be a topological group.

Let X be a T_1 space and also a group in which $qy=px$, p, q integers, $x \in X$, has a unique solution $y \in X$. By rx, r rational, $x \in X$, one means the unique solution y of the corresponding equation $qy=px$. The space X is said to satisfy condition A' if given any open set O and point $p \in O$ there exists an open set U containing p such that if $p \in rU$, then $rU \subset O$. It is shown that if the scalar product rx is continuous in x for fixed r and continuous in r for fixed x and if X satisfies condition A', then rx is continuous in r and x simultaneously except at θ. Finally, if X is a metrizable linear space over the rationals with invariant metric $d(x, y)$ such that (1) for any $x \neq \theta$, $x \in X$, there exists an $r>0$ and unique $z \in X$ such that $x=rz$ and $d(z, \theta)=1$; (2) if $m>n$, where m and n are positive integers, then $d(mx, \theta)>d(nx, \theta)$ for all $x \in X$ and (3) there exist positive constants A and B such that if $d(z, \theta)=1$ then $Ar \leqq d(rz, \theta) \leqq Br$ for all $r>0$, then X can be imbedded isometrically in a Banach space.

W. R. Utz (Columbia, Mo.).

Kyle, R. H. Branched covering spaces and the quadratic forms of links. Ann. of Math. (2) **59**, 539–548 (1954).

Two symmetric integral matrices A and B are defined to be of the same family if A can be transformed into B by a finite sequence of elementary operations Q_i $(i=1, 2, 3)$, where $Q_1: A \to T'AT$, T integral and unimodular,

$$Q_2: \ A \to \begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad \text{and} \quad Q_3: \ A \to \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

This definition of "family" is suggested from knot theory, where a knot k of arbitrary finite multiplicity defines by means of a regular projection a matrix A, and a semilinear isotopy of k leaves the family of A invariant. Thus every knot k defines a family F_k invariant under s (semi-linear)-isotopy of k. A group with linking or V-group is a finite Abelian group G together with a primitive bilinear symmetric mapping V of G into the rationals mod 1. Any nonsingular integral matrix A defines a V-group $G(A^{-1})$. This group is generated by $g_1, \cdots, g_n$, subject to relations $\sum_{i=1} a_{ij} g_j = 0$, and $V(g_i, g_j) = (A^{-1})_{ij}$ mod 1.

It is now shown that every family contains a nonsingular symmetric matrix, and that two symmetric nonsingular matrices are in the same family if and only if they have isomorphic V-groups. The V group of a family is therefore well defined, as the V-group of any nonsingular member of the family. The principal theorem of the first part of the paper shows the V-group of the family of a knot to be isomorphic to the torsion group of the second branched covering $\mathfrak{M}_2$ of k. A corollary of this generalization of Seifert's theorem for knots of multiplicity 1 is that the family of k is actually an isotopy invariant of k (since the torsion group of $\mathfrak{M}_2$ is known to be determined by the group of the knot). As is pointed out in the paper, this fact also follows from the general equivalence of s-isotopy and isotopy proved by Moise [Ann. of Math. (2) **59**, 159–170 (1954); these Rev. **15**, 889].

The second part of the paper deals with number-theoretical invariants of a family and examples of their applicability.

R. Bott (Ann Arbor, Mich.).

Torres, G., and Fox, R. H. Dual presentations of the group of a knot. Ann. of Math. (2) **59**, 211–218 (1954).

This paper brings a new conceptual proof of the symmetry of the Alexander polynomial of a multiple knot, originally due to Torres [Ann. of Math. (2) **57**, 57–89 (1953); these Rev. **14**, 574]. Assuming familiarity with the papers on the free differential calculus [ibid. **57**, 547–560 (1953); **59**, 196–210 (1954); these Rev. **14**, 843; **15**, 931] of R. H. Fox, the authors state the main results of this paper as follows.

Two presentations $(x; \eta)$ and $(y; \xi)$ of G are called dual if

$$x_i \equiv y_i^{-1}, \mod (\chi\phi),$$

$$\frac{\partial \eta_j}{\partial x_i}(x_i - 1) = \frac{\partial \xi_i}{\partial y_j}(y_j - 1), \mod (\chi\phi).$$

Here $\varphi: JX \to JG$ $[JY \to JG]$ is induced by the natural projection of the free group generated by the generators (x_i) $[(y_i)]$ onto G, while $\chi: JG \to JH$ is induced by the projection $G \to G/[G, G]$. The bar operation is an involution of JG defined by $\sum a_i f_i = \sum a_i f_i^{-1}$. Theorem: The group of any tame knot in 3-space has a dual pair of presentations. Corollary 1: The group of a tame knot in 3-space has an Alexander matrix that is equivalent to the conjugate of its transpose; consequently, the elementary ideals $E_1, E_2, \cdots$ are invariant under conjugation. Corollary 2: The Alexander polynomial Δ of a tame single knot has the property $\Delta(t) = t^m \Delta(t^{-1})$, where $m \equiv 0 \bmod 2$. Corollary 3: The Alexander polynomial Δ of a tame knot of multiplicity $\mu>1$ in 3-space has the property

$$\Delta(t_1, \cdots, t_\mu) = (-1) t_1^{m_1 - 1} \cdots t_\mu^{m_\mu - 1} \Delta(t_1^{-1}, \cdots, t_\mu^{-1}),$$

where $m_i \equiv$ the linking number of the ith component with the sum of the other components (mod 2). *R. M. Bott.*

Chang, S. C. Note on homotopy types and cohomology systems. Acad. Sinica Science Record **4**, 217–222 (1951). (Chinese summary)

The author gives new proofs of two of the main theorems of a paper by J. H. C. Whitehead [Ann. Soc. Polon. Math. **21**, 176–186 (1949); these Rev. **11**, 48]. *W. S. Massey.*

Liao, S. D. On the theory of obstructions of fiber bundles. Ann. of Math. (2) **60**, 146–191 (1954).

The author studies obstructions to the existence of cross-sections in fiber bundles. The main results concern the theory of the primary obstruction in a bundle whose fiber is $(m-1)$-connected and a formula for the secondary obstruction in a sphere bundle.

For the results on the primary obstruction let $\phi: E \to B$ be a bundle projection with $(m-1)$-connected fiber F $(m \geqq 2)$ and the base space B a connected complex. For simplicity in describing the results below we assume that the fundamental group of B acts trivially on $\pi_m(F)$ (though this assumption is not made at this stage in the paper). Let $\kappa \in H^m(F; \pi_m(F))$ denote the primary obstruction to contracting F to a point and let $i: F \to E$ denote the inclusion map. To every cross-section f over B^{m+1} there is associated a unique element $\theta(f) \in H^m(E; \pi_m(F))$ characterized by the conditions $f^* \theta(f) = 0$ and $i^* \theta(f) = \kappa$. The element $\theta(f)$ has the property that two cross-sections f, f' over B^{m+1} are homotopic over B^m if and only if $\theta(f) = \theta(f')$ and for any $u \in H^m(E; \pi_m(F))$ such that $i^* u = \kappa$ there is a cross-section f over B^{m+1} with $\theta(f) = u$. Furthermore, if B_0 is a non-vacuous subcomplex of B and f_0 is a cross-section over

B_0, then ϕ^* is an isomorphism of $H^{m+1}(B, B_0; \pi_m(F))$ into $H^{m+1}(E, \phi^{-1}(B_0); \pi_m(F))$ and

$$\phi^*\Omega = -\delta\theta(f_0),$$

where $\Omega \in H^{m+1}(B, B_0; \pi_m(F))$ is the primary obstruction to extending f over B and

$$\delta: H^m(\phi^{-1}(B_0); \pi_m(F)) \to H^{m+1}(E, \phi^{-1}(B_0); \pi_m(F))$$

is the coboundary homomorphism.

For the results on the secondary obstructions on sphere bundles we use the same notation as before except that the fiber is S^m. We assume that the primary obstruction vanishes so that there exist cross-sections f over B^{m+1}. Let $Z^{m+2}(f) \in H^{m+2}(B; \pi_{m+1}(S^m))$ be the secondary obstruction to the extension of $f|B^m$ (i.e. $Z^{m+2}(f)$ vanishes if and only if there is a cross-section over B^{m+2} which agrees with f over B^m). Let $\beta: H^n(E; G) \to H^{n-m}(E; G)$ be the homomorphism called "integration over the fiber" and let

$$\mathrm{Sq}^2: H^m(E; \pi_m(S^m)) \to H^{m+2}(E; \pi_{m+1}(S^m)) \quad \text{for} \quad m > 2$$

denote the Steenrod square. Then ϕ^* is an isomorphism of $H^n(B; G)$ into $H^n(E; G)$ for every n and

$$\phi^*Z^{m+2}(f) = \theta(f) \smile \phi^*\beta\,\mathrm{Sq}^2(f) + \mathrm{Sq}^2(f) \quad \text{for} \quad m > 2,$$
$$\phi^*Z^4(f) = \theta(f) \smile \phi^*\beta(\theta(f) \smile \theta(f)) - \theta(f) \smile \theta(f) \quad \text{for} \quad m = 2.$$

These formulas can be simplified for the case where the structure group of the bundle is the rotation group.

The proofs use the machinery of spectral sequences, and for the results on the secondary obstruction, the original bundle is partially imbedded in the bundle of symmetric squares of S^m over B. *E. H. Spanier* (Chicago, Ill.).

Wada, Hidekazu. Note on the fibering of an $(n-1)$-connected space by spheres. Proc. Japan Acad. **29**, 415–417 (1953).

Let $n > 2$. Assume X is an $(n-1)$-connected fiber bundle with fiber a $(k-1)$-sphere S^{k-1} $(n > k > 1)$ and base space Y. Denote by ΔY the minimal dimension of all CW-complexes dominating Y, and let M equal $2k-2$, or $2k-3$, according as the suspension $\pi_{2k-3}(S^{k-1}) \to \pi_{2k-2}(S^k)$ is or is not an isomorphism. It is shown that, if one requires $\Delta Y \leqq \min(n-1, M)$, then (a) Y must belong to the homotopy type of S^k and (b) there can be no such fibering with $2k \leqq n$. It follows that there is no fibering of any Euclidean space by $(k-1)$-spheres which has $\Delta Y \leqq M$. *J. Dugundji* (Los Angeles, Calif.).

GEOMETRY

Andersson, J. Sur l'équation cubique. Mathesis **63**, 102–104 (1954).

Goormaghtigh, R. Sur une affinité complexe. Mathesis **63**, nos. 3–4–5, supplément, 1–28 (1954).

For nearly four decades the author has been studying the transformation, in rectangular cartesian coordinates,

$$x = x_1, \quad y = iy_1 \quad (i^2 = -1),$$

and used it in a considerable number of his published papers. In the present monograph the author presents his method and his results, with some additions, in a systematic form, grouped in four chapters, under the headings: I. Propriétés fondamentales (pp. 1–3); II. Quelques applications à la géométrie plane (pp. 3–13); III. Affines de courbes et de transformations géométriques (pp. 14–22); IV. Transformation des propriétés infinitésimales des courbes (pp. 22–28).

N. A. Court (Norman, Okla.).

Mikusiński, J. G. Sur le parquetage du plan par des polygones. Colloquium Math. **3**, 14–18 (1954).

Let the real Euclidean plane be filled by a map of f_3 triangles, f_4 quadrangles, f_5 pentagons, and infinitely many hexagons, in such a way that each vertex belongs to three of the polygons. By considering first the dual map, consisting of triangles, the author proves that

$$3f_3 + 2f_4 + f_5 \leqq 6.$$

He remarks that the plane can be filled with an infinity of heptagons. His drawing of this is a Euclidean representation of the regular map $\{7, 3\}$ in the hyperbolic plane.

H. S. M. Coxeter (Toronto, Ont.).

Coxeter, H. S. M., Longuet-Higgins, M. S., and Miller, J. C. P. Uniform polyhedra. Philos. Trans. Roy. Soc. London. Ser. A. **246**, 401–450 (6 plates) (1954).

Das Ziel der Arbeit ist, alle gleichförmigen Polyeder (d.s. Polyeder mit regulären Begrenzungsflächen und äquivalenten Ecken) aufzustellen. Neben den fünf Platonischen Körpern, den 13 Archimedischen Körpern, den vier regulären Sternkörpern von Kepler und der unendlichen Schar von Prismen und Antiprismen finden die Autoren deren 53. Der Beweis, dass es deren keine weiteren gibt, wird nicht erbracht. Alle bis auf eines dieser Polyeder können durch eine Konstruktion von Wythoff hergeleitet werden: Durch Spiegelung an den Seiten eines passenden sphärischen Dreiecks. In Tabellen werden die metrischen Eigenschaften und die Eckfiguren zu jedem Polyeder angegeben, von jedem Polyeder wird eine Zeichnung und das photographische Bild eines Drahtmodelles beigegeben.

J. J. Burckhardt (Zürich).

Obláth, Richárd. Une nouvelle démonstration du théorème de M. Bauer sur la théorie des constructions géométriques. Mat. Lapok **4**, 108–112 (1953). (Hungarian. Russian and French summaries)

Godeaux, Lucien. Remarque sur les homographies cycliques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40**, 569–573 (1954).

Briquet, André. Sur les cubiques de Mac Cay. Mathesis **63**, 105–111 (1954).

Stefánsson, Sigurkarl. Two loci connected with the equilateral hyperbola. Nordisk Mat. Tidskr. **2**, 44–52, 80 (1954). (Danish. English summary)

Facciotti, G. Una generalizzazione della iperbole. Period. Mat. (4) **32**, 57–69 (1954).

***Haack, Wolfgang. Darstellende Geometrie. I. Die wichtigsten Darstellungsmethoden. Grund- und Aufriss ebenflächiger Körper.** Sammlung Göschen Bd 142. Walter de Gruyter & Co., Berlin, 1954. 110 pp. DM 2.40.

This first volume of a work planned in three volumes starts with a short historical introduction, after which the

principal projection methods are explained. Stress in the following exposition is on orthographic projection, which is carried as far as the penetration of polyhedra. The final section deals with affinities and is mainly concerned with the projection of a circle into an ellipse. *D. J. Struik.*

***Haack, Wolfgang. Darstellende Geometrie. II. Körper mit krummen Begrenzungsflächen. Kotierte Projektionen.** Sammlung Göschen Bd 143. Walter de Gruyter & Co., Berlin, 1954. 129 pp. DM 2.40.

This second volume continues with the application of orthographic projection to cylinder, cone, and sphere and their interpenetrations. Then follow surfaces of rotation and helicoidal surfaces; the final chapter, of 31 pages, explains numbered projections (kotierte Projektionen). Interesting is the derivation of the properties of conic sections from the property of two congruent cones of parallel axes to intersect in a conic section, due to H. Herrmann. The chapter on numbered projections serves as an introduction to K. Bartel, Kotierte Projektionen [Teubner, Leipzig-Berlin, 1933]. The figures in both first and second volume are remarkably clear, despite the small size of the booklets. The third volume is expected to stress the immediate visual impression of the drawing. *D. J. Struik* (Cambridge, Mass.).

Wyszecki, G. Invariance of insidedness in projective transformations of the Maxwell triangle. J. Opt. Soc. Amer. 44, 524–529 (1954).

Devidé, Vladimir. Über ein Modell der euklidischen Geometrie. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 8, 241–246 (1953). (Serbo-Croatian summary)

The author observes that the image of a Euclidean plane under a polarity of its projective extension is a model of a Euclidean plane, with lines being "points", etc.
A. J. Hoffman (Washington, D. C.).

Szász, Paul. Über die Trigonometrie des Poincaréschen Kreismodells der hyperbolischen ebenen Geometrie. Acta Math. Acad. Sci. Hungar. 5, 29–34 (1954). (Russian summary)

Poincaré's mapping of the hyperbolic plane onto the interior of a circle is used to derive the formulae of hyperbolic trigonometry. *E. Lukacs* (Washington, D. C.).

Zassenhaus, Hans. What is an angle? Amer. Math. Monthly 61, 369–378 (1954).

The author presents an axiomatic treatment of the notion of angle, which is satisfied by the model of an ordered pair of open rays with the angular space between them. The axioms involve the addition (not always defined) of angles, and the notion of positive and negative angles. With the help of a continuity axiom, it is shown that suitably defined equivalence classes of formal sums of angles are isomorphic to the additive group of reals, and the choice of isomorphism determines the measure of angles. *A. J. Hoffman.*

Maruyama, Takaharu. Some investigations in mathematical cartography. J. Gakugei, Tokushima Univ. (Nat. Sci.) 4, 48–53 (1954).

A slight extension of the generalized conical projection described by König and Weise in their Mathematische Grundlagen der höheren Geodäsie und Kartographie [1. Bd., Springer, Berlin, 1951; these Rev. 13, 157] is examined. The transformation $A=\exp f(M)$, where M is König's complex longitude, is considered under the condition that the projection distortion and its derivatives are zero at the origin. *N. A. Hall* (Minneapolis, Minn.).

Convex Domains, Extremal Problems, Integral Geometry

Tatarkiewicz, Krzysztof. Quelques remarques sur la convexité des sphères. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 6 (1952), 19–30 (1954). (Polish and Russian summaries)

Elementary observations concerning convex subsets of surfaces of spheres in normed linear spaces. *V. L. Klee.*

Davis, Chandler. Remarks on a previous paper. Michigan Math. J. 2, 23–25 (1954).

The "previous paper" of the title appeared in same J. 1, 163–168 (1953); these Rev. 14, 1055. Following a suggestion of R. Bott (also noted, in part, by the reviewer), shorter proofs based on standard duality theorems are given for the following two results (slightly generalized from the originals). (1) If $U\subset E^k$ and $V\subset E^n$ are polyhedral convex cones satisfying $D\{U\}=E^k$ and $D\{V^+\}=E^n$, then $AU=V\cap D\{AU\}$ implies $A'V^+=U^+\cap D\{A'V^+\}$. (2) Under the same hypotheses, $A'V^+$ is affine-equivalent to the geometric polar of AU. Here, $D\{S\}$ denotes the smallest subspace containing a cone S, S^+ is the positive polar of S, A is an $n\times k$ matrix, and prime denotes transpose. As an application, a simple construction is given to show that every pointed convex polyhedral cone is affine-equivalent to the intersection of the positive orthant with a linear subspace. By standard methods, this is restated: Every convex polytope is congruent to a section of a simplex. The dual statements are also made. *H. W. Kuhn.*

Radziszewski, Konstanty. Sur un problème extrémal relatif aux figures inscrites et circonscrites aux figures convexes. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 6 (1952), 5–18 (1954). (Polish and Russian summaries)

The author establishes the following results. (I) Given a plane convex figure of area $A>0$, there exists an inscribed rectangle of area $\geqq A/2$. (II) Given in 3-space a convex body of volume $V>0$, there exists a circumscribed parallelepiped of volume $\leqq 6V$; moreover, if the body has a plane of symmetry, there exists an inscribed parallelepiped of volume ~~$\geqq 3V/9$~~ $\geqq aV/9$. *L. C. Young* (Madison, Wis.).

Kubota, Tadahiko, and Hemmi, Denzaburo. Some problems of minima concerning the oval. J. Math. Soc. Japan 5, 372–389 (1953).

If $b(\phi)$ is the breadth of an oval K in the direction ϕ, a "breadth curve" B of K is a central symmetric oval with the breath $2b(\phi)$; it may be defined as the envelope of the lines t whose distance to a fixed point is equal to the breadth of K parallel to t. Let $P_1P_2\cdots P_6$ be a convex hexagon inscribed in B with the same centre of symmetry as B and such that $P_1P_4=2P_2P_5$. The triangle formed by the sides P_1P_3, P_3P_4, P_5P_6 bordered by the corresponding arcs of K, is an oval having the breadth of K in every direction; it is called the "asymmetric oval" determined by the "base hexagon" $P_1P_2\cdots P_6$. The authors prove the following fundamental theorem: Let S be a hexagon of the largest

area among all the base hexagons inscribed in the breadth curve B of K. Then, the asymmetric oval determined by S and B is of the smallest area among all ovals with the breadth of K in every direction. From this theorem the two following inequalities are deduced.

(a) If the diameter D and the smallest breadth Δ of an oval of area F satisfy the relation $D \geqq \Delta \geqq \frac{1}{2}D\sqrt{3}$, then the following inequality holds:

$$F \geqq 3\Delta\left\{(D^2-\Delta^2)^{\frac{1}{2}}+\Delta\left(\sin^{-1}\frac{\Delta}{D}-\frac{\pi}{3}\right)\right\}-\frac{\sqrt{3}}{2}D^2.$$

The equality occurs when and only when the oval is an asymmetric curve whose base hexagon is a regular hexagon of sides D and whose breadth curve is a minimum oval enclosing the base hexagon and a concentric circle with radius Δ.

(b) If Δ and the perimeter L of an oval satisfy the relation $\pi\Delta \leqq L \leqq 2\Delta\sqrt{3}$, then the following inequality holds:

$$2F \geqq \Delta L - 3^{\frac{1}{2}}\Delta^2 \sec^2\theta,$$

where θ is the root of $\tan\theta-\theta=(L-\pi\Delta)/6\Delta$ in the interval $0 \leqq \theta \leqq \pi/6$. The equality occurs when and only when the oval is an asymmetric curve whose base hexagon is a regular hexagon of sides $\Delta\sec\theta$ and whose breadth curve is a minimum oval enclosing the base hexagon and the concentric circle with radius Δ. *L. A. Santaló* (Buenos Aires).

Hadwiger, H. Von der Zerlegung der Kugel in kleinere Teile. Gaz. Mat., Lisboa 15, no. 57, 1–3 (1954). (German and Portuguese)

Let the n-dimensional solid sphere of unit diameter be decomposed into m subsets $S_1, S_2, \cdots, S_m$. If $m=n$, Borsuk showed that diam $S_i=1$ for at least one index i [Verh. Internat. Math.-Kongresses, Zürich, 1932, vol. 2, Füssli, Zürich-Leipzig, 1933, pp. 192–193]. In this note $m=n+1$ and the problem (proposed by Knaster [loc. cit., p. 193]) of determining the smallest number D_n such that diam $S_i \leqq D_n$ for $i=1, 2, \cdots, n+1$ is solved. For $n \geqq 2$ it is shown that

$$D_n \geqq \left[\frac{1}{2}+\frac{1}{2}\left(\frac{n-1}{2n}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

with the equality sign holding for $n=2$, and $n=3$.

L. M. Blumenthal (Leiden).

Nef, Walter. Konvexe Räume. Arch. Math. 4, 216–221 (1953).

The author introduces a notion of independence in the abstract convex spaces of Kneser [Arch. Math. 3, 198–206 (1952); these Rev. 14, 478]. A set S is said to be independent if the expression of every point of the convex hull of S as a convex combination of a finite number of points of S is unique. Every convex space has at least one basis and the following decomposition theorem holds. Let $\{\xi_i\}$ be a basis for a convex space K over a commutative positive domain P. For each $x \in K$, there exists a unique set of elements of P, $\alpha>0$, $\alpha_i \geqq 0$, $\alpha_i' \geqq 0$ (with only a finite number of α_i, $\alpha_i' \neq 0$) such that $\sum\alpha_i=\alpha+\sum\alpha_i'=1$ and $\alpha x+\sum\alpha_i'\xi_i=\sum\alpha_i\xi_i$. As applications, a simple proof of the embedding theorem of Kneser [loc. cit.] is given, and analogous results of H. Hadwiger [Comment. Math. Helv. 24, 204–218 (1950); these Rev. 12, 526] on the decomposition of polyhedra are derived. *H. W. Kuhn* (Bryn Mawr, Pa.).

Algebraic Geometry

Tanturri, Giuseppe. Una particolare quartica piana con catene di flessi. Boll. Un. Mat. Ital. (3) 9, 143–146 (1954).

Godeaux, Lucien. Sur la représentation plane de la surface cubique et sur une transformation birationnelle qui s'en déduit. Mathesis 63, 97–102 (1954).

Heymans, P. The representation of algebraic doubly infinite line systems. J. London Math. Soc. 29, 309–318 (1954).

Si l'on représente les droites d'un espace S_r à r dimensions par les points d'une variété V_{2r-2}, par exemple la variété de Grassmann habituelle, alors les droites d'un système algébrique G, ∞^2, se représentent par les points d'une surface algébrique G^*. Supposons que G soit d'ordre n, de classe s, et d'index 1 sur la variété Γ (d'ordre n aussi) qui est recouverte par ses droites; alors, si l'on considère: la courbe g section de Γ par un espace S_{r-2}, les ∞^1 groupes de s droites de G existant dans les ∞^1 hyperplans qui passent par S_{r-2}, et les ∞^{r-3} groupes de n droites de G qui s'appuient aux différents S_{r-3} de S_{r-2}, on reconnait que la section hyperplane g^* de la surface G^* dont les points représentent les droites de G s'appuyant à S_{r-2} (et par là à g) contient deux séries linéaires g_s^1 et g_n^{r-2} dont la somme minimale est contenue dans la série caractéristique découpée sur g^* par les hyperplans de l'espace ambient. L'auteur démontre que cette propriété est aussi suffisante; c'est-a-dire, si une section hyperplane f d'une surface normale F, d'ordre $N=n+s$, contient deux séries linéaires g_s^1 et g_n^{r-3}, simples et tout points fixes, qui se trouvent dans les conditions énoncées plus haut, alors il existe dans un espace S_r, avec $r \geqq 4$, un système ∞^2 de droites, de classe s, ordre n et index 1, dont les droites se peuvent représenter par les points d'une surface homographique ou à F ou à une projection de F. On applique ensuite ce théorème à l'étude des surfaces directrices de G, des points singuliers de G, des autres systèmes de droites qui peuvent exister sur la même variété Γ; tout cela est illustré en particulier dans le cas où F est la surface normale F^6 de Del Pezzo de première espèce. En général, on peut classifier les systèmes G en partant d'une classification des surfaces algébriques et des séries g_s^1 existant sur une de leurs sections hyperplanes. *E. G. Togliatti* (Gênes).

Marchionna, Ermanno. Sulle varietà aritmeticamente normali. Atti Sem. Mat. Fis. Univ. Modena 6 (1951–52), 45–56 (1953).

Two theorems which give conditions on a plane projection of a non-singular irreducible space curve C which are necessary and sufficient for C to be arithmetically normal are presented. Extensions to curves in higher-dimensional space and to varieties of higher dimension are also mentioned.

H. T. Muhly (Iowa City, Iowa).

Nollet, Louis. La régularité des axes du premier groupe de torsion des surfaces algébriques. Bull. Soc. Roy. Sci. Liège 23, 115–124 (1954).

L'auteur considère en premier lieu, dans un espace projectif complexe S, un ensemble H de m homographies H_i ($i=1, \cdots, m$) cycliques et à deux à deux permutables. Si $U_{i1}\cdots U_{ip_i}$ sont les espaces unis de H_i, les points unis pour toutes les homographies de H sont les points communs aux espaces $U_{1h_1}\cdots U_{mh_m}$ pour tous les choix possibles des nombres $h_i \leqq p_i$; ces points forment un ensemble de sous-

espaces de S, que l'on appellera les axes de H, qui sont disjoints deux à deux, et qui déterminent l'espace ambient S pourvu que leur nombre soit supérieur à $p_1 \cdots p_m(1-p^{-1})$, où p est le plus grand des entiers p_i. Ces propriétés s'appliquent, en particulier, à un groupe abélien G, d'ordre fini, d'homographies; alors H peut être un système canonique de générateurs de G; on trouve que les axes de H (ou G) déterminent l'espace S toutes les fois que leur nombre est plus grand que $t(1-1/t_m)$, où t est l'ordre de G et t_m la plus grande période d'une homographie de H. Tout cela trouve une application dans le problème de la dimension des systèmes pseudocanoniques d'une surface algébrique F douée de torsion. L'auteur suppose que le genre arithmétique p_a de F ne soit pas négatif. Si G est le premier groupe de torsion de F, son ordre t (>1) est le diviseur de Severi de F; et G est isomorphe au module des diviseurs algébriques de zéro. On trouve que G est isomorphe à un groupe de t homographies dans un espace de dimension $t(p_a+1)+q-2$, où $q=p_g-p_a$; et que les systèmes pseudocanoniques de F donnent lieu aux axes de G. La conclusion que l'on en tire est que les systèmes pseudocanoniques de F sont presque partout réguliers de dimension p_a et toujours réguliers de dimension p_a si F est régulière. *E. Togliatti* (Gênes).

Matsusaka, Teruhisa. On the algebraic construction of the Picard variety. I. Jap. J. Math. **21** (1951), 217–235 (1952).

Matsusaka, Teruhisa. On the algebraic construction of the Picard variety. II. Jap. J. Math. **22** (1952), 51–62 (1953).

The author gives a detailed account of a construction of the Picard variety P of an algebraic variety V, a construction which was sketched in a previous paper [Proc. Japan Acad. **28**, 5–8 (1952); these Rev. **14**, 81; the notations and definitions used here are the same as in that review]. Paper I contains the following additional features: a) Criteria for linear equivalence, using the generic 1-section C, are proved both for arbitrary divisors and for divisors in $G_a(V)$. b) The construction of P runs as follows: take the variety W having as generic point the Chow-point M of the associated variety $T(X)$ of the complete linear system determined by a generic element X of the algebraic system S; there is a normal law of composition on W, and, by a theorem of A. Weil, W is birationally equivalent to an abelian variety P. c) There exists a rational mapping ψ of V into P such that P is generated by $\psi(V)$, every element of P being the sum of at most q ($=\dim(P)$) elements of $\psi(V)$.

In paper II another construction of the Picard variety is given, parallel to the (unpublished) method of W. L. Chow for constructing the Jacobian variety of a curve. Every "sufficiently large" maximal algebraic family S of positive divisors of V has the following properties: every $D \in G_a(V)$ is linearly equivalent to a divisor of the form $X'-X''$ ($X', X'' \in S$); all complete linear systems contained in S have the same dimension. Such a family S exists which contains a divisor X_0 rational over the smallest field of definition k of V. Then the variety W defined as in paper I (cf. b) above) admits a normal law of composition (defined by $X \sim X'+X''-X_0$), for which the sum of any two points of W is uniquely determined. This does not necessarily imply that this law of composition is "everywhere defined" (in the technical sense), but the corresponding law of composition on a derived normal model P of W is, and P is an abelian variety since it is a projective variety. This variety P is isomorphic to the Picard variety defined in paper I. The following important additional property is proved: there exists a common field of definition k' of P and V such that, if a is a point of P, there exists a representative divisor $D \in G_a(V)$ of a which is rational over $k'(a)$; this property, together with its converse proved in paper I, determines uniquely the Picard variety P.

Paper II ends with a construction of the Albanese variety A of V. One uses the fact that an abelian variety generated by V (more precisely, by a rational image of V) is also generated by a generic 1-section of V, and therefore has a bounded dimension. Let B be an abelian variety of maximum dimension q generated by V, and let f denote the corresponding mapping of V into B. If (x_i) $(1 \leqq i \leqq q)$ are independent generic points of V over a suitable common field of definition K, the point $z=\sum_{i=1}^{q} f(x_i)$ is a generic point of B, and the field of symmetric functions $K(x_1, \cdots, x_q)$, is an algebraic extension of finite degree $d(B, f)$ of $K(z)$. If, among the pairs (B, f), we take a pair (A, h) for which $d(A, h)$ is minimum, we get for A the Albanese variety of V and for h the canonical mapping of V into A: for every rational mapping g of V into an abelian variety C, there exists a homomorphism-translation g' of A into C such that $g=g' \circ h$. *P. Samuel* (Clermont-Ferrand).

Matsusaka, Teruhisa. Some theorems on Abelian Varieties. Nat. Sci. Rep. Ochanomizu Univ. **4**, 22–35 (1953).

The author proves that every abstract abelian Variety may be imbedded in a projective space. More precisely, if U is an abstract Variety having a normal law of composition [cf. Weil, Variétés abéliennes et courbes algébriques, Hermann, Paris, 1948; these Rev. **10**, 621], if k is a field of definition for U and for that law, and if U is birationally equivalent to some abstract abelian Variety, then it is birationally equivalent over k to an abelian variety in a projective space.

As is nowadays inevitable, the paper begins by lemmas showing that certain properties are true almost everywhere. When (W) is an algebraic system of positive cycles, then almost all cycles of (W) are nonsingular varieties provided one of them is. If p is a fixed projection, it is regular at every point of almost all cycles of (W), provided it is regular at every point of some cycle of W. If (W) and (A) are algebraic systems of positive cycles, and if a generic element A of (A) is an abelian variety generated by a generic element W of (W) [cf. Matsusaka, Nat. Sci. Rep. Ochanomizu Univ. **3**, 1–4 (1952); these Rev. **14**, 201], then, for almost all specializations (W', A') of (W, A), A' is an abelian variety generated by W'. Another lemma (using Zariski's main theorem on birational transformations) shows that, if a normal projective variety U admits a normal law of composition $(x, y) \to x \cdot y$ such that, for all x', y' in U, $x \cdot y$, $x \cdot y^{-1}$ and $x^{-1}y$ have only a finite number of specializations over $(x, y) \to (x', y')$, then U is an abelian variety.

Let now V be a projective normal variety, and let P and A denote its Picard and its Albanese varieties; in previous papers [reviewed above] the author proved that P is a projective abelian variety, and that A is an abstract abelian Variety. He constructs a "Poincaré family" for V, i.e. an algebraic family $(X(a))$ $(a \in A)$ of positive divisors on V, parametrized by A, and such that every divisor $X(a)$ is linearly equivalent to at most a finite number of divisors of the form $X(b)$ $(b \in A)$. This shows that A and P are isogeneous abelian varieties [cf. Weil, loc. cit.]. Their common dimension is called the irregularity of V. In an algebraic family (V) of positive cycles whose generic ele-

ment is a variety, almost all cycles of (V) are varieties whose irregularities are greater than that of V.

Then the proof of the main result runs as follows. If the abstract Variety U with normal law of composition is birationally equivalent with some abstract abelian Variety A, A is the Albanese Variety of U. As the Albanese Variety of U is obviously an absolute birational invariant of U, A is isogeneous to some projective abelian variety P, e.g., with the Picard variety of any projective normal model of U. Let g be a homomorphism of A onto P. One can construct a normal projective variety B, birationally equivalent to A, and such that the function g' transformed from g is everywhere defined on B, and such that $g'^{-1}(x)$ is a finite set of points for every x in P. Then the normal law of composition on B deduced from the addition on A satisfies the requirements of a lemma described above, and makes out of B an abelian variety. The fact that B and the birational transformation T of U onto B may be construed to be defined over k is then proved by a construction of "symmetric functions", considering essentially the conjugates of T over k. *P. Samuel* (Clermont-Ferrand).

Northcott, D. G. On the local cone of a point on an algebraic variety. J. London Math. Soc. 29, 326–333 (1954).

Soient V une variété algébrique de l'espace affine A_n, P un point de V (qu'on suppose placé à l'origine), $\mathfrak{o}$ l'anneau local de P sur V, $\mathfrak{m}$ son idéal maximal. L'anneau gradué associé $\sum_{s=0}^{\infty}\mathfrak{m}^s/\mathfrak{m}^{s+1}$ est un quotient de l'anneau de polynomes $K[X_1, \cdots, X_n]$ par un idéal homogène $\mathfrak{a}$; l'ensemble algébrique défini par celui ci est appelé le cone des tangentes à V en P et noté $C(P/V)$ [cf. J. Igusa, Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. 27, 189–201 (1952); ces Rev. 14, 580]. En utilisant les spécialisations, affines et projectives, prolongeant l'homomorphisme canonique de $\mathfrak{o}$ sur $\mathfrak{o}/\mathfrak{m}$, on montre que les génératrices de $C(P/V)$ sont les sécantes limites de V passant par P, c.à.d., en notant $\bar{P}$ un point générique de V, les spécialisations de la droite $P\bar{P}$ prolongeant $\bar{P}\to P$. Si V et V' sont en correspondance birationnelle et si celle ci est birégulière en P et P', les cones $C(P/V)$ et $C(P'/V')$ sont affinement équivalents. Enfin si P est un point commun à deux variétés V et W de dimensions complémentaires, et si $C(P/V)$ et $C(P/W)$ n'ont pas de génératrice commune, alors P est une composante de $V\cap W$, et la multiplicité d'intersection $i(P; V\cdot W)$ est égale au produit des multiplicités de P sur V et W, ce qui généralise le critère de multiplicité 1. Un grand usage est fait de la théorie des réductions d'idéaux [Northcott et Rees, Proc. Cambridge Philos. Soc. 50, 145–158 (1954); ces Rev. 15, 596]. *P. Samuel* (Clermont-Ferrand).

Zariski, O. Le problème de la réduction des singularités d'une variété algébrique. Bull. Sci. Math. (2) 78, 31–40 (1954).

The purpose of this paper is to report the progress made by the author towards the general solution of the problem of the reduction of the singularities of an algebraic variety. The only case considered is that in which the general field is the field of complex numbers (though the methods are valid for any field of characteristic zero), and the proofs are only sketched.

Let V_r be any pure r-dimensional variety, and suppose that it is immersed in the direct product $F_{r+1}=F_r\times S_1$ of a non-singular variety F_r and a straight line, such that (1) the projection of V_r on F_r covers F_r, and (2) each point of F_r is the projection of a finite number n of points of V_r. The locus of points of F_r which are the projections of less than n distinct points of V_r constitute the critical variety V_{r-1} of V_r in F_r. The reduction problem is reduced to the fundamental lemma (as yet unproved): If the singularities of V_{r-1} are only normal crossings (i.e. such that that through each singular point Q of V_{r-1} there pass exactly two components of V_{r-1} which are non-singular and do not touch at Q), then there exists a birational transformation ϕ of F_{r+1} with the properties: (a) ϕ transforms F_{r+1} into a non-singular variety F'; (b) ϕ^{-1} is regular on F'; (c) the locus of points on F' at which ϕ^{-1} is not biregular is a pure variety of dimension r; (d) the only singularities of the variety $\phi\{V_r\}$, which consists of the proper transforms of the components of V_r and the exceptional varieties arising from the fundamental points of ϕ, have only normal crossings.

If $F_r=F_{r-1}\times S_1$, $F_{r-1}=F_{r-2}\times S_1, \cdots$ and the critical variety V_{r-1} in F_r satisfies (1) and (2), the critical variety V_{r-2} of V_{r-1} in F_{r-1} satisfies (1) and (2), $\cdots$, V_r is said to be regularly immersed in F_{r+1}. By showing that any immersed variety of dimension r can be regularly immersed in F_n and using the fundamental lemma, it is explained by an induction argument how this leads to a solution of the general reduction problem.

A sketch is given of the proof of the fundamental lemma in the case in which V_r is irreducible and analytically irreducible at every point, and it is suggested that the reduction problem can probably be reduced to the following unsolved problem: If V_r is an irreducible variety immersed in a non-singular variety G_{r+1}, does there always exist a birational transformation ϕ of G_{r+1} satisfying (a), (b), (c) (above) such that the transform of V_r by ϕ is analytically irreducible at every point? *W. V. D. Hodge.*

Differential Geometry

***Haantjes, J. Inleiding tot de Differentiaalmeetkunde. [Introduction to differential geometry.]** P. Noordhoff N. V., Groningen-Djakarta, 1954. v+173 pp. 7.50 florins; bound, 9.50 florins.

This attractive book is an introduction to the classical differential geometry of curves and surfaces in euclidean three-dimensional space with the aid of the vector and the tensor calculus. In its seven chapters we find not only a clear and comprehensive exposition of such subjects as the Frenet formulas, the Gauss-Codazzi equations and the Gauss-Bonnet theorem, but also a short but adequate treatment of the elements of the analysis of vectors and tensors. These methods are introduced only at the place where they are needed for the geometrical demonstrations. Some special features are a sketch of Bonnet's existence theorem for surfaces with given first and second fundamental tensors and a derivation of the Gauss-Bonnet theorem based on the consideration of triangles of which two sides are geodesic. There are more than a hundred problems, with indications for their solution at the end, and thirty-five good illustrations. *D. J. Struik* (Cambridge, Mass.).

Kneser, Hellmuth. Monoton gekrümmte ebene Kurven. Arch. Math. 5, 77–80 (1954).

Suppose the plane curve $C=\mathfrak{x}(t)$ is twice differentiable for $a<t<b$ and $d\mathfrak{x}/dt$ never vanishes. If the curvature $\kappa(t)$ is of bounded variation, then it is automatically continuous. Using this observation, the following results are proved without assuming the existence of the third derivative of $\mathfrak{x}(t)$. Let $\kappa(t)$ be positive everywhere. (i) Let $a<t_0<b$. Let

$\kappa(t)$ be monotonic; $\kappa(t)\neq\kappa(t_0)$ $(t_0<t<b)$. Then the evolute of C has a right tangent at $t=t_0$ and the latter is equal to the normal of C at $\Gamma(t_0)$. (ii) Let $\kappa(t)$ be of bounded variation. Then the length of an arc of the evolute of C is equal to the total variation of the radius of curvature $\kappa^{-1}(t)$ on the corresponding arc of C. Thus if $\kappa(t)$ is monotonic, that length is equal to the absolute value of the difference of the radii of curvature at the end-points of the arc of C. The author points out that (i) does not remain valid if $\kappa(t)$ is not monotonic but only of bounded variation. *P. Scherk.*

Balliccioni, A. Application des coordonnées barycentriques à des propriétés relatives à la courbure des coniques et des cubiques. Mathesis 63, 112–117 (1954).

***Godeaux, Lucien. Quadriques et coniques de Moutard.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 152–161. Edizioni Cremonese, Roma, 1954. 4000 Lire.

Wunderlich, Walter. Eine bemerkenswerte Fokaleigenschaft der D-Kurven von Kegeln 2. Grades. Monatsh. Math. 58, 57–62 (1954).

A D-curve of a surface F is a curve on F whose osculating spheres are tangent spheres of F. If F is a quadric, the tangents of a D-curve are parallel to the generators of a certain cone of order two. The planes which intersect this cone in circles will also yield circular intersections of F. Let F be a cone of degree two. Then the above properties are sufficient for the determination of its D-curves. They also imply: Every D-curve on F can be interpreted as a loxodrome in six ways, i.e., it intersects the pencil of the planes through the generators of F and any one of the six real or imaginary focal axes of F at a constant angle.
P. Scherk (Saskatoon, Sask.).

Wunderlich, Walter. Über die ebenen Loxodromen. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. 162, 285–292 (1953).

A loxodrome is a curve meeting the planes of an axial family at a fixed angle. If a loxodrome is a plane curve, reflection in this plane shows it is also a loxodrome with respect to a second family of planes. It is then doubly a loxodrome. The axes of the two families are called axes of the loxodrome. The author considers plane loxodromes whose axes are parallel, intersecting, or isotropic, respectively. It is shown that in the general case of intersecting axes, the plane loxodrome is actually quadruply a loxodrome. The four axes form a tetrahedral angle with isotropic faces. *S. B. Jackson* (College Park, Md.).

Niče, Vilko. Über die isotropen Strahlenpaare 2. Art der Strahlenkongruenzen 1. Ordnung 3., 2. und 1. Klasse. Hrvatsko Prirodoslovno Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 8, 293–296 (1953). (Serbo-Croatian summary)

Datt, Uma, and Rambehari. Some theorems on normal rectilinear congruences. Proc. Nat. Acad. Sci. India. Sect. A. 20, 205–208 (1951).

Les auteurs reprennent, par la méthode tensorielle, la démonstration d'une proposition de Ram Behari relative à l'existence de ∞^2 surfaces réglées d'une congruence rectiligne quelconque (∞^1 si la congruence est normale) pour lesquelles les quadriques osculatrices sont équilatères, ainsi que l'établissement du caractère isothermique des représentations sphériques des surfaces réglées principales ou distributrices de la congruence des normales à une surface minima. *P. Vincensini* (Marseille).

Jeger, M. Zur projektiven Differentialgeometrie der ebenen Kurven-3-Gewebe. Math. Z. 60, 112–119 (1954).

The system of curves in the plane which satisfy equations analogous to the usual geodesic equations but with arbitrary functions Γ^i_{jk} instead of the usual Christoffel symbols is called a quasigeodesic system. The system does not uniquely define the set of functions Γ^i_{jk}, but determines a whole class of such sets of functions all yielding the same quasigeodesic system. It is known that the curves of a web belong to infinitely many quasigeodesic systems. The goal of this paper is the investigation of the projective invariants of a web with respect to the quasigeodesic systems to which its curves belong, that is, quantities depending only on the web and one of the quasigeodesic systems containing it. It is found for example that there is a tensor, called the embedding tensor, differing for different quasigeodesic systems, whose vanishing characterizes the so-called cross-ratio system. *S. B. Jackson* (College Park, Md.).

Pinl, M. J. On the theory of half minimal surfaces. Pakistan J. Sci. Res. 3, 101–105 (1951).

It is known that minimal surfaces in Euclidean 3-space contain two coupled isotropic systems of imprimitivity, that is, they are in two ways surfaces of translation of isotropic curves. In this paper a study is made of translation surfaces for which only one of the families of generating curves is isotropic. These are called half-minimal surfaces. Among the translation surfaces, the minimal surfaces are characterized by the second-order equation $H=0$, where H is the mean curvature. Similarly the half-minimal surfaces are characterized by a third-order differential equation, which can be expressed as a first-order differential equation in H and the Gaussian curvature K. *S. B. Jackson.*

Löbell, Frank. Kriterien für die Integrabilität von Richtungsübertragungen in Flächen. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1953, 141–148 (1954).

A displacement of direction (Richtungsübertragung) is a law by which a vector tangent to a surface is carried into another vector tangent to the surface as its contact point is moved. Equivalently it can be considered to define a rigid motion of the tangent plane to a surface as the contact point is varied. The author has shown [Jber. Deutsch. Math. Verein. 55, Abt. 1, 89–119 (1952); these Rev. 13, 984] that such a law of displacement is uniquely determined by a scalar function ψ of the general directed line element of the surface. The displacement is called integrable if the final position of a vector which is moved is independent of the path of motion. Previous derivations of criteria for integrability have been based on the theorem that for this it is necessary that the function ψ be linear in a certain sense. A new derivation of the criteria is here given based on the variation of the curvilinear integral of a line-element function. The validity of the linearity condition is not assumed but comes out incidentally in the discussion.
S. B. Jackson (College Park, Md.).

Lense, Josef. Zum Einbettungssatz der Differentialgeometrie. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1953, 69–75 (1954).

Consider the differential quadratic form $g_{\nu\mu}dx^\nu dx^\mu$ where the $g_{\nu\mu}$ are complex analytic functions of the complex vari-

ables x^α and where the indices run from 1 to n. The embedding problem is to embed this space in a complex Euclidean space of some dimension N. This is equivalent analytically to finding N functions ξ_k, $k=1, \cdots, N$, of the x^α so that $\sum_{k=1}^{N} d\xi_k^2 = g_{\alpha\nu}dx^\alpha dx^\nu$ is an identity. The embedding theorem is the assertion that this is possible if $N=n(n+1)/2$. This has been established by Cartan [Ann. Soc. Polon. Math. 6, 1–7 (1928)] for the case when the rank r of the matrix $(g_{\alpha\nu})$ is n. Janet [ibid. 5, 38–43 (1927)] considered the theorem without restriction on r and proved it by induction, provided a certain determinant does not vanish identically. The author considers here the cases $n=1$ and $n=2$ when this determinant does vanish identically. The case $n=1$ is trivial while for $n=2$ it is shown that the condition characterizes the isotropic planes. A proof covering all cases is not known. *S. B. Jackson* (College Park, Md.).

Lavrent'ev, M. A. Stability in Liouville's theorem. Doklady Akad. Nauk SSSR (N.S.) **95**, 925–926 (1954). (Russian)

It is a classical result of Liouville that the only conformal transformations in Euclidean 3-space consist of the rigid motions, similarity transformations, inversions, or a combination of these three. The author considers a homeomorphism of a region D of 3-space onto another region Δ, where the mapping has a positive Jacobian and satisfies a Hölder condition. Let $p=p(x, y, z)$ denote the ratio of the largest and smallest axes of the infinitesimal ellipsoid in Δ which is the image under the mapping of an infinitesimal sphere in D. The author finds that, for p near 1, the mapping of D onto Δ is "fairly close" to one of those permitted by Liouville's theorem. *A. J. Lohwater* (Ann Arbor, Mich.).

Ōtsuki, Tominosuke. A construction of closed surfaces of negative curvature in E^4. Math. J. Okayama Univ. **3**, 95–108 (1954).

The paper is concerned with the differential geometry of closed two-dimensional surfaces in four-dimensional Euclidean space. It is proved that the surface generated by the circles of constant radius whose centers lie on a closed curve in a two-dimensional plane E^2 and whose planes are perpendicular to E^2 has everywhere zero Gaussian curvature. A construction is given of a closed surface of negative Gaussian curvature, whose genus is 7. *S. Chern* (Chicago, Ill.).

Kručkovič, G. I. Classification of three-dimensional Riemannian spaces according to groups of motions. Uspehi Matem. Nauk (N.S.) **9**, no. 1(59), 3–40 (1954). (Russian)

The problem under discussion is simplified by the fact that a Riemannian V_3 can not admit a group of motions of more than six parameters, that is, the case of spaces of constant curvature. Neither can a V_3 admit a complete group of five parameters. Spaces admitting a group G_1 are well known so that the questions remaining are concerned with groups G_2, G_3, and G_4. All these groups have been classified by Lie so that the problem reduces to the solution of Killing's equations for the two G_2's, nine G_3's, and eight G_4's. The author solves the problem completely and obtains in each case the group generators and the fundamental quadratic form in a suitably chosen co-ordinate system.
M. S. Knebelman (Pullman, Wash.).

Gaffney, Matthew P. A special Stokes's theorem for complete Riemannian manifolds. Ann. of Math. (2) **60**, 140–145 (1954).

The absolute value $|\alpha|$ of a p-form α at a point of a Riemannian manifold is defined to be $[*(\alpha^*\alpha)]^{1/2}$, and α is said to be L_k if $|\alpha|$ is L_k. The theorem proved in this paper states that if M is an orientable complete Riemannian manifold of dimension n whose metric is of class C^2, and if γ is an $(n-1)$-form of class C^1 on M such that γ and $d\gamma$ are both L_1, then $\int_M d\gamma = 0$. *W. V. D. Hodge.*

Ledger, A. J. Harmonic homogeneous spaces of Lie groups. J. London Math. Soc. **29**, 345–347 (1954).

It is proved that if a homogeneous space E of a Lie group G admits a positive definite Riemannian metric invariant under G, and if an isotropy subgroup of G is locally spherically transitive, then E (with the above metric) is completely harmonic. This theorem is applied to the compact two-point homogeneous spaces considered by H. C. Wang [Ann. of Math. (2) **55**, 177–191 (1952); these Rev. **13**, 863] and it is deduced that the real, complex and quaternionic projective spaces and the Cayley projective plane admit positive definite harmonic Riemannian metrics. Results are also extended to certain non-compact spaces.
A. G. Walker (Liverpool).

Katsurada, Yoshie. On the extended connection parameters in a space with affine connection and in a Riemannian space. J. Fac. Sci. Hokkaido Univ. Ser. I. **12**, 17–28 (1951).

Es sei A_n ein affinzusammenhängender Raum mit Übertragungsparametern Γ^a_{bc}. Stellt $x^i(t)$ eine differenzierbare Kurve dar, dann erhält man durch $(\alpha-\beta-\gamma)$-fache Ableitung die erweiterten Übertragungsparameter

$$\Gamma^{aa}_{\beta b\gamma c}=\begin{cases}\dfrac{\alpha!}{\beta!\gamma!(\alpha-\beta-\gamma)!}\Gamma^a_{bc}{}^{(\alpha+\beta+\gamma)} & \text{für } \beta+\gamma\leqq\alpha,\\ 0 & \text{für } \beta+\gamma>\alpha.\end{cases}$$

Mit Hilfe dieser erweiterten Übertragungsparameter soll ein erweiterter kovarianter Differentiationsprozess hergeleitet werden, der folgende Eigenschaft besitzt; wird derselbe auf einen erweiterten Tensor (Extensor) angewandt, dann soll diese Operation für die Komponenten, die den gewöhnlichen Tensor darstellen mit der gewöhnlichen kovarianten Ableitung zusammenfallen. Im Falle eines Riemannschen Raumes wird die entsprechende Fragestellung behandelt. Es ergibt sich dabei, dass die erweiterten Übertragungsparameter mit den erweiterten Christoffelschen Symbolen identisch sind. Letztere werden aus den erweiterten metrischen Grundtensor

$$g^{M}_{\alpha\alpha\beta b}=\frac{M!}{\alpha!\beta!(M-\alpha-\beta)!}g_{ab}^{(M-\alpha-\beta)},\quad \alpha,\beta=0,\cdots,M,$$

hergeleitet, der dadurch entsteht, dass man g_{ab} längs der Kurve $x^i(t)$ $(M-\alpha-\beta)$-mal differenziert. *O. Varga.*

Katsurada, Yoshie. Non-holonomic system in a space of higher order. II. On the theory of extensors on the subspace. J. Fac. Sci. Hokkaido Univ. Ser. I. **12**, 29–41 (1951).

Bekanntlich sind die Extensoren geometrische Objekte denen Transformationsgesetz hinsichtlich erweiterten Punkttransformationen definiert sind. Dadurch sind die Extensoren das natürliche analytische Instrument in der Theorie der Räume "höherer Ordnung" deren Grundelement das Linienelement höherer Ordnung ist.

Es sei

$$x^i=x^i(u^{i'}), \quad i=1, \cdots, n; \qquad i'=1, \cdots, m, \quad m \leqq n,$$

die Darstellung einer Untermannigfaltigkeit eines n-dimensionalen Raumes. Sind die $u^{i'}$ Funktionen eines Parameters, dann treten, falls nach demselben fortgesetzt bis zur M-ten Ordnung differenziert wird, sowohl die höheren Ableitungen der x^i als der $u^{i'}$ bis zur M-ten Ordnung auf. Die Mannigfaltigkeit, die so durch $x^i, x^{ii'}, \cdots, x^{(M)i}$ bestimmt ist, heisst eine erweiterte Untermannigfaltigkeit (exsurface) M-ter Ordnung, und $u^{i'}, \cdots, u^{(M)i'}$ die erweiterten Parameter. Gestützt auf die von Verfasserin entwickelte Theorie anholonomer Koordinatensysteme in Räumen höherer Ordnung [J. Fac. Sci. Hokkaido Univ. Ser. I. 11, 190–217 (1950); diese Rev. 13, 779] wird jetzt die Theorie der Extensoren in erweiterten Untermannigfaltigkeiten dargestellt. Verfasserin kann sich dabei auf eine Reihe von Ergebnissen von Kawaguchi [ibid. 9, 1–152 (1940); diese Rev. 2, 22] stützen. Falls eine Mannigfaltigkeit höherer Ordnung mit affinem Zusammenhang vorliegt, wird eine entsprechende D-Symbolik entwickelt. Dieselbe ermöglicht die Einführung des erweiterten Euler-Schoutenschen Krümmungstensors, und damit die Entwicklung der gesamten Krümmungstheorie. *O. Varga* (Debrecen).

Katsurada, Yoshie. Specialization of the theory of a space of higher order. I. On the extended non-holonomic system. Jap. J. Math. 21 (1951), 237–248 (1952).

Verfasserin will die Theorie der Räume von Linienelementen höherer Ordnung als eine Theorie der erweiterten geometrischen Objekte eines Punktraumes darstellen. In vorliegender Arbeit sollen erweiterte anholonome Koordinatensysteme untersucht werden.

Es sei K_n ein n-dimensionaler Punktraum und $K_n^{(M)}$ der Raum der Linienelemente M-ter Ordnung. Das anholonome System im K_n sei durch die Vektoren $\lambda^i_{i'}$ ($i'=1, \cdots, n$) und die Anholonomiegruppe durch

$$\lambda^i_{i'}=N^{i'}_{i}\lambda^i_{i'}, \quad |N^{i'}_{i}| \neq 0, \tag{1}$$

bestimmt. Längs der differenzierbaren Kurve $x^i(t)$ möge $\lambda^i_{i'}$ $(\alpha-\tau)$-mal differenziert werden. Die sich so ergebenden Grössen

$$\lambda^{\alpha i}_{\tau i'}=\begin{cases}\binom{\alpha}{\gamma}\lambda^{i(\alpha-\gamma)}_{i'} & \text{für } \alpha \geqq \gamma \\ 0 & \text{für } \alpha<\gamma\end{cases}$$

bestimmen das erweiterte anholonome System. Entsprechend erhält man aus (1) die erweiterte anholonome Gruppe

$$\lambda^{\alpha i}_{\tau i'}=\sum_{\beta=\tau}^{\alpha} N^{\beta\alpha'}_{\tau i'}\lambda^{\alpha i}_{\beta\alpha'} \quad (\alpha, \tau=0, \cdots, M),$$

wobei

$$N^{\beta\alpha'}_{\tau i'}=\binom{\beta}{\gamma}N^{\alpha'(\beta-\tau)}_{i'}$$

und

$$|N^{\beta\alpha'}_{\tau i'}| \neq 0$$

gilt. Verfasserin untersucht wie sich die in ihren beiden früheren Arbeiten [J. Fac. Sci. Hokkaido Univ. Ser. I. 11, 190–217 (1950); Tensor (N.S.) 1, 60–66 (1951); diese Rev. 13, 384, 779] angegebenen Anholonomieoperatoren und Anholonomiegrössen in den erweiterten anholonomen Koordinatensystem ausdrücken. Es werden dann die speziellen Fälle untersucht, in denen der K_n entweder ein symmetrischer affinzusammenhängender, oder ein Riemannscher Raum ist. In diesen Fällen wird insbesondere der Zusammenhang der erweiterten Übertragungsparameter mit den Übertragungsparametern des anholonomen Systems untersucht. Die entsprechenden Zusammenhänge werden auch für die Theorie der Untermannigfaltigkeiten ermittelt. Ein $K_n^{(M)}$, in dem sämtliche geometrische Objekte durch Erweiterung von Objekten eines Punktraumes K_n entstehen, wird von Verfasserin als erweiterter K_n bezeichnet. Die beiden obigen Fälle sind demnach erweiterte affinzusammenhängende, bezw. erweiterte, Riemannsche Räume. *O. Varga* (Debrecen).

Katsurada, Yoshie. Specialization of the theory of a space of higher order. II. On the extended Lie derivative. Tensor (N.S.) 2, 15–26 (1952).

In einem Raum von Linienelementen höherer Ordnung, der die Erweiterung eines Punktraumes im Sinne der vorangehend referierten Arbeit ist, wird der Begriff der erweiterten Lieschen Ableitung eingeführt. Ist

$$\bar{x}^i=x^i+\xi^i(x)\delta t$$

eine infinitesimale Transformation eines n-dimensionalen Punktraumes K_n und ist $\xi^{(\alpha)i}$ der kontravariant erweiterte Erweiterungstensor, der durch Ableitung von ξ^i längs der Kurve $x^i(t)$ hinsichtlich t gewonnen wird, dann soll

$$\bar{x}^{(\alpha)i}=x^{(\alpha)i}+\xi^{(\alpha)i}(x, \cdots, x^{(\alpha)})\delta t$$

als die erweiterte infinitesimale Transformation des Raumes $K_n^{(M)}$ bezeichnet werden. Die Liesche Ableitung eines geometrischen Objektes $f(x, \cdots, x^{(M)})$ im $K_n^{(M)}$ ist dann durch

$$Df=Xf\cdot\delta t=f(\bar{x}, \bar{x}^{(1)}, \cdots, \bar{x}^{(M)})-\bar{f}(\bar{x}, \bar{x}^{(1)}, \cdots, \bar{x}^{(M)}) \tag{1}$$

definiert, wobei $\bar{f}$ das Objekt in den $\bar{x}^{(\alpha)i}$ Koordinaten bedeutet. Entsprechend der Definition (1) wird die Liesche Ableitung für Skalare, erweiterte Vektoren beider Arten und erweiterte Tensoren angegeben. Die Formeln werden auch für diejenigen Fälle hergeleitet in denen der K_n eine lineare Übertragung Γ^i_{jk} besitzt. In diesem Falle werden auch die Lieschen Ableitungen der erweiterten Übertragungsparameter bestimmt. Verfasserin bestimmt schliesslich den Zusammenhang zwischen den Lieschen Ableitungen eines geometrischen Objektes in K_n und der Lieschen Ableitung des erweiterten geometrischen Objektes in $K_n^{(M)}$. Ist z.B. v^i ein kontravarianter Vektor in K_n und $v^{(\gamma)i}$ der erweiterte Tensor in $K_n^{(M)}$, so gilt

$$Xv^{(\gamma)i}=(Xv^i)^{(\gamma)}.$$

Ähnliche einfache Zusammenhänge bestehen auch für beliebige Tensoren.

Im Falle der Raum einen Affinzusammenhang besitzt, oder ein Riemannscher ist, ermöglichen diese Zusammenhänge die Bestimmung von Räumen die erweiterte Affinitäten bezw. erweiterte Bewegungsgruppen besitzen. *O. Varga* (Debrecen).

Katsurada, Yoshie. Specialization of the theory of a space of higher order. III. On the extended projective and conformal invariants. Tensor (N.S.) 2, 181–188 (1952).

Verfasserin gibt eine Begründung der T. Y. Thomasschen Theorie der projektiv und konform zusammenhängenden Räume, für Räume von Linienelementen höherer Ordnung. Entsprechend dem Gang dieser Theorie werden zunächst mit Hilfe erweiterter Übertragungsparameter erweiterte Bahnkurven (expath) definiert. Sodann werden die erweiterten bahntreuen Transformationen der Übertragungsparameter angegeben. Die hinsichtlich dieser Transformationen invarianten geometrischen Objekte stellen genau die erweiterten Thomasschen projektiven Übertragungspara-

meter dar. Wie im klassischen Falle können nun projektive Koordinaten eingeführt werden. Für eine gewisse beschränkte Transformationsgruppe dieser Veränderlichen können nun die erweiterten Thomasschen projektiven Übertragungsparameter so ergänzt werden, dass sie die erweiterten Übertragungsparameter in einem erweiterten affinen Raum darstellen, dessen Dimension um eins grösser ist als die des ursprünglichen projektiven Raumes. Dieser Raum ist die affine Darstellung des projektiven Raumes. Durch diese Entwicklungen ist die projektive Theorie völlig begründet, da ja jede projektive eine nunmehr theoretisch feststellbare Invariante der affinen Darstellung ist. Verfasserin fügt in diesen Rahmen noch erweiterte projektive Jakobische Tensoren ein, und bestimmt die erweiterten projektiven Krümmungstensoren, die speziell die verallgemeinerten Weylschen Tensoren enthalten.

Bei der Thomasschen Begründung der konformen Theorie spielt ausser der affinen Darstellung A^*_{n+1} des konformen Raumes C_n noch eine Matrix, die durch Ränderung der zur entsprechenden Transformationsgruppe des A^*_{n+1} gehörigen Funktionalmatrix entsteht, eine grundlegende Rolle. Verfasserin betrachtet die Elemente dieser Thomasschen Matrix als ein anholonomes System, das sie entsprechend erweitert. Mit Hilfe der so gewonnene Objekten lässt sich dann das Transformationsgesetz der erweiterten konformen Übertragungsparameter definieren. Es ermöglicht sich nun die Einführung einer erweiterten invarianten konformen Differentiation und die Einführung von konformen Krümmungstensoren. *O. Varga* (Debrecen).

Katsurada, Yoshie. On the parallel displacement of arc. Tensor (N.S.) **2**, 85–88 (1952).

Gegeben sei ein n-dimensionaler affinzusammenhängender Raum L_n und in demselben eine m-dimensionale Untermannigfaltigkeit S

$$(1) \qquad x^i = x^i(u^{i'}), \quad i = 1, 2, \cdots, n, \quad i' = 1, 2, \cdots, m.$$

Eine die Untermannigfaltigkeit S im Punkte P_0 in M-ter Ordnung berührende ebenfalls m-dimensionale Untermannigfaltigkeit $S_0^{(M)}$ sei dadurch bestimmt, dass man in der Taylorentwicklung von (1) nur Glieder bis zur Ableitung M-ter Ordnung beibehält. Eine durch P_0 gehende auf S liegende Kurve sei durch

$$(2) \qquad u^{i'} = u^{i'}(t)$$

gegeben und auf $S_0^{(M)}$ die gleichfalls durch P_0 gehende Kurve $C_0^{(M)}$ dadurch, dass man in der Taylorentwicklung von (2) nur Glieder mit Ableitungen bis zur M-ten Ordnung berücksichtigt. C_0 und $C_0^{(M)}$ berühren sich in M-ter Ordnung. Es wird nun auf S ein schlichtes Kurvenfeld angenommen, und ferner in zwei benachbarten Punkten P_0 und P_1 von S, die auf den Scharkurven C_0 und C_1 liegen, im Sinne obiger Angaben, die Flächen $S_0^{(M)}$ und $S_1^{(M)}$ mit den Kurven $C_0^{(M)}$ bezw. $C_1^{(M)}$ konstruiert. Für die Punkte der beiden Kurven existieren die Linienelemente M-ter Ordnung $u^{(\alpha+1)i'}$ $(\alpha = 0, \cdots, M-1)$. Entsprechend existieren die erweiterten Übertragungsparameter $\Gamma^{\alpha i'}_{\beta j', \nu}$. Falls nun für gleiche Werte des Parameters die Linienelemente von $C_0^{(M)}$ und $C_1^{(M)}$ im Sinne der Craigschen erweiterten kovarianten Ableitung parallel sind, bezeichnet Verfasserin die Bogen $C_0^{(M)}$ und $C_1^{(M)}$ als parallel. Die erweiterten Übertragungsparameter können daher so interpretiert werden, dass sie eine Abbildung induzieren die eine Berührung M-ter Ordnung invariant lässt. Falls der Grenzübergang $M \to \infty$ möglich ist, werden die Bogen C_0 und C_1 als parallel bezeichnet. Verfasserin bezeichnet eine Übertragung von der Klasse, bezw. als regulär, falls für den Index α der erweiterten Parameter $\Gamma^{\alpha i'}_{\beta j', \nu}$, $\alpha = 0, \cdots, R$, bezw. $\alpha = 0, \cdots, \infty$, gilt. *O. Varga* (Debrecen).

Katsurada, Yoshie. On the parallel displacement of subspaces in an affinely connected space. Tensor (N.S.) **3**, 1–12 (1953).

Die in der voranstehenden Arbeit für affinzusammenhängende Mannigfaltigkeit erklärte Parallelübertragung von Bogen, wird jetzt für Riemannsche Räume untersucht, ausserdem wird eine Parallelübertragung für Flächen erklärt.

In Riemannschen Räumen ergibt sich, dass bei der erwähnten Parallelübertragung, die durch

$$P^i_M = \sum_{\alpha,\beta=0}^{M-1} \xi_{\alpha\beta i} x^{(\alpha+1)i} x^{(\beta+1)i}$$

erklärte Länge von Linienelementen $x^{(\alpha+1)}$, M-ter Ordnung invariant bleibt. Insbesondere ergibt sich dass für die parallelen Bogen $C_0^{(M)}$ und $\bar{C}_0^{(M)}$ die Länge von Tangentenvektoren ungeändert bleibt, falls Grössen höherer als M-ter Ordnung des Parameters $(t - t_0)$ [siehe das voranstehende Referat] vernachlässigt werden. Der Grenzübergang zeigt, dass die Parallelübertragung der Bogen eine isometrische Abbildung ist.

Um die Parallelübertragung von m-dimensionalen Flächen zu erklären, wird die aus einer solchen Fläche durch Erweiterung M-ter Ordnung erhaltene Mannigfaltigkeit von Flächenelementen M-ter Ordnung $S_n(M; m)$ gebildet. Auf einer solchen Mannigfaltigkeit existieren nach A. Kawaguchi [Monatsh. Math. Phys. **48**, 329–339 (1939); diese Rev. 1, 87] verallgemeinerte erweiterte Tensoren und für dieselben gibt es nach T. Ohkubo [J. Fac. Sci. Hokkaido Imp. Univ. Ser. I. **11**, 1–37 (1946); diese Rev. **10**, 404] Übertragungsparameter.

Verfasserin betrachtet ein Feld von m-dimensionalen Flächen, die das in Betracht kommende Gebiet schlicht bedecken. In einem solchen Falle kann mit Hilfe eines entsprechenden kovarianten Differentials, die Parallelübertragung der erwähnten Kawaguchischen Tensoren durchgeführt werden. Zu der durch den Punkt P_0 gehenden m-dimensionale Mannigfaltigkeit S kann, mit Hilfe ihrer Taylorschen Entwicklung (entsprechend der in der voranstehend referierten Arbeit) eine sie in P_0 in M-ter Ordnung berührende Fläche $S_0^{(M)}$ konstruiert werden. Für die durch den Nachbarpunkt $\bar{P}_0$ gehende m-dimensionale Fläche $\bar{S}$ kann auf dieselbe Weise eine sie in $\bar{P}_0$ in M-ter Ordnung berührende Fläche bestimmt werden. Die erwähnten Mannigfaltigkeiten sind dabei auf dieselben Parameter bezogen und die Koeffizienten in den Darstellungen von $S_0^{(M)}$ und $\bar{S}_0^{(M)}$ sind gerade die Bestimmungszahlen von Flächenelementen M-ter Ordnung. Sind nun die Flächenelemente M-ter Ordnung für gleiche Parameterwerte parallel, dann bezeichnet Verfasserin die Flächen $S_0^{(M)}$ und $\bar{S}_0^{(M)}$ als parallelübertragen. Falls der Grenzübergang $M \to \infty$ möglich ist, dann sind S und $\bar{S}$ parallelübertragen. *O. Varga* (Debrecen).

Katsurada, Yoshie. A geometrical consideration of the Craig excovariant differential. Tensor (N.S.) **2**, 80–84 (1952).

Im vierdimensionalen euklidischen Raum sei eine zweidimensionale Fläche C_2 gegeben. Auf C_2 sei eine schlichte Kurvenschar gegeben. Verfasserin bildet die Erweiterung zweiter Ordnung $C_2^{(2)}$ von C_2. Die Flächenparameter von

C_2 seien $u^{i'}$ ($i'=1, 2$) und $u^{(\alpha+1)i'}$ ($\alpha=0, 1$) ihre Erweiterungen. Sind nun in zwei benachbarten Punkten P_0 und P_1 von C_2, die auf je einer Scharkurve C_0 bezw. C_1 liegen, die Linienelemente zweiter Ordnung $(u_0^{(1)i'}, u_0^{(2)i'})$ bezw. $(u_1^{(1)i'}, u_1^{(2)i'})$ gegeben, dann beweist Verfasserin, dass der Parallelismus dieser Linienelemente, im Sinne der erweiterten Craigschen kovarianten Ableitung, die Gleichheit der ersten Krümmungen von C_0 und C_1 in den Punkten P_0 und P_1 bedeutet. Die Behauptung ergibt sich aus der Darstellung

$$\kappa_1^2 = \sum_{\alpha,\beta=0}^{1} g_{\alpha i'\beta j'} u^{(\alpha+1)i'} u^{(\beta+1)j'}$$

der ersten Krümmung einer Kurve auf C_2 und daraus, dass für den Fundamentaltensor $g_{\alpha i'\beta j'}$ von $C_2^{(2)}$ die Craigsche Ableitung verschwindet. Daraus folgt nun, dass eine Berührung zweiter Ordnung von Kurven bei dieser Übertragung invariant ist. Die Übertragung wird demgemäss als erste Krümmungsübertragung bezeichnet. Falls die Fläche eine Ebene ist, leifert die Übertragung eine Translation von Kurvenbogen. *O. Varga* (Debrecen).

Katsurada, Yoshie. On the intrinsic derivative in the non-holonomic exsurface. J. Fac. Sci. Hokkaido Univ. Ser. I. 12, 157–162 (1953).

Zu Grunde gelegt sei ein Raum $K_n^{(M)}$ von Linienelementen M-ter Ordnung. Ist $(x, x', \cdots, x^{(M)})$ ein Linienelement M-ter Ordnung, und $v^{(\alpha)i}$ ein erweiterter Tensor, dann definiert Verfasserin eine anholonome erweiterte Fläche N_n' durch die n Pfaffschen Gleichungen

$$\sum_{\alpha=0}^{M} \lambda_{\alpha i}^{\alpha'}(x, x', \cdots, x^{(M)}) dv^{(\alpha)i} = 0 \quad (\alpha'=1, \cdots, n).$$

Ist $v^{\alpha i}$ ein erweiterter Tensor vom Range R in $K_n^{(M)}$, dann bestimmen die durch

$$\binom{M}{R} v^{\alpha'} = \sum_{\alpha=M-R}^{M} \binom{\alpha}{M-R} \lambda_{\alpha i}^{\alpha'} v^{(\alpha-M+R)i}$$

definierte Grössen $v^{\alpha'}$ die Vektorkomponenten von $v^{\alpha i}$ in N_n' Eine Verallgemeinerung der Craigschen "intrinsic" Ableitung [Bull. Amer. Math. Soc. **53**, 332–342 (1947); diese Rev. **8**, 491] für $v^{\alpha'}$ ist durch

$$\binom{M}{R} \frac{\delta^R v^{\alpha'}}{dt^R} = \sum_{\alpha=M-R}^{M} \binom{\alpha}{M-R} \lambda_{\alpha i}^{\alpha'} v^{(\alpha-M+R)i}$$

definiert. Verfasserin zeigt, dass sich auch in diesem allgemeineren Falle diese Ableitung durch Überschiebungen von solchen erweiterten Tensoren darstellen lässt, die im wesentlichen durch $\lambda_{\alpha j}^{\alpha'}$ und deren Erweiterung bestimmt sind. *O. Varga* (Debrecen).

Ohkubo, Takeo. Homogeneous contact transformations in a generalized space K_n. Kumamoto J. Sci. Ser. A. 1, no. 1, 27–40 (1952).

Verf. betrachtet eine Mannigfaltigkeit K_n von Hyperflächenelementen (ξ^α, π_α). Die Punktkoordinaten sind mit ξ^α diejenigen der Hyperebenen mit π_α bezeichnet. Ist $C(\xi, \pi)$ die charakteristische Funktion einer einparametrigen Gruppe von Berührungstransformationen, dann wird durch

$$\gamma^{\alpha\beta} = \partial^2\Pi/\partial\pi_\alpha\partial\pi_\beta, \quad 2\Pi = C^2$$

eine Metrik in K_n eingeführt. Von den $\gamma^{\alpha\beta}$ ausgehend, leitet Verf. die Cartanschen Übertragungsparameter $C_\alpha{}^{\beta\gamma}$ und $\Gamma^\gamma{}_{\alpha\gamma}$ her. Indem auf K_n eine Berührungstransformation ausgeübt wird, erhält Verf. eine Mannigfaltigkeit T_n von Hyperflächenelementen. Im T_n kann Verf. auf entsprechende Weise wie Eisenhart [Ann. of Math. (2) **49**, 227–254 (1948); diese Rev. **9**, 380] eine Übertragung einführen. Unter besonderen Bedingungen kann T_n mit K_n zusammenfallen. *O. Varga* (Debrecen).

Ide, Saburo. On the connections in higher order spaces. Tensor (N.S.) **3**, 84–90 (1954).

W. Wirtinger hat eine Übertragung für solche Mannigfaltigkeiten X_n angegeben in denen in jedem Punkt einem Bündel von kontravarianten Vektoren η^α ein Bündel von kovarianten Vektoren v_α mit der Nebenbedingung $\eta^\alpha v_\alpha = 0$ zugeordnet ist. Die Übertragung ist dadurch festgelegt, dass bei der Variation δ_t des Punktes x^α für $\eta^\alpha dv_\alpha$ die Bedingungen einer Berührungstransformation erfüllt sind. Verf. gibt eine direkte Verallgemeinerung dieser Wirtingerschen Übertragung für Kawaguchische Räume von Linienelementen höherer Ordnung. Die Verallgemeinerung besteht darin, dass jetzt die Variation nicht von einem Punkt, sondern von dem Linienelement höherer Ordnung abhängt. Die so gewonnene Übertragung wird dann für den Fall modifiziert in dem der vorher willkürliche Vektor v_α nunmehr von x^α und η^α abhängt. Verf. studiert dann den Zusammenhang mit den von Kawaguchi eingeführten Übertragungen in gewissen speziellen Kawaguchischen Räumen. *O. Varga* (Debrecen).

***Reeb, Georges. Sur certains problèmes relatifs aux variétés presque-symplectiques et systèmes dynamiques.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 104–106. Edizioni Cremonese, Roma, 1954. 4000 Lire.

Boothby, William M. Some fundamental formulas for Hermitian manifolds with non-vanishing torsion. Amer. J. Math. **76**, 509–534 (1954).

This paper develops systematically the tensor calculus on a Hermitian manifold M, using Chern's technique of the fibre bundle B of all orthogonal frames of M. This first part of the paper gives an account of the general theory, and is not new, though the previous literature is somewhat scattered. The paper goes on to obtain some of the fundamental formulae of Hermitian geometry, such as the Bianchi identities, and to study the forms $d\theta$, $*\theta$, $\delta\theta$ obtained from a form θ on M when referred to B. Of particular interest is the fact that some of the known results in the Kähler case, i.e. the case in which the torsion forms $\Omega^i = A^i{}_{j\bar{k}}\omega^j\omega^k$ vanish, can be extended to the cases in which $A^i{}_{j\bar{k}l} = 0$ or $A^k{}_{j\bar{k}} = 0$. *W. V. D. Hodge* (Cambridge, England).

***Yano, K., and Bochner, S. Curvature and Betti numbers.** Annals of Mathematics Studies, No. 32. Princeton University Press, Princeton, N. J., 1953. ix+190 pp. $3.00.

Dans ce livre, les auteurs se proposent de donner un exposé systématique des travaux de géométrie différentielle globale qui dérivent d'une idée remarquable de Bochner [Bull. Amer. Math. Soc. **52**, 776–797 (1946); ces Rev. **8**, 230] et qui sont dus principalement à Bochner lui-même, à Lichnerowicz [C. R. Acad. Sci. Paris **226**, 1678–1680 (1948); Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 2, Amer. Math. Soc., Providence, R. I., 1952, pp. 216–223; ces Rev. **9**, 618; **13**, 492] et à Yano [Ann. of Math. (2) **55**, 38–45, 328–347 (1952); ces Rev. **13**, 689; **14**, 88]. Le principe de la méthode est le suivant: soit V_n une variété riemannienne compacte orientable dont on se propose d'atteindre les nombres de Betti par l'intermédiaire des

formes harmoniques ϕ. Si ϕ^2 est le carré scalaire local d'une telle forme, on peut déduire des conditions d'harmonicité et des identités de Ricci une expression du laplacien $\Delta[\phi^2]$, en fonction du carré de la dérivée covariante de ϕ et d'un scalaire faisant intervenir la courbure de la variété et les composantes de ϕ. Sous certains hypothèses sur la courbure, on a $\Delta[\phi^2] \geqq 0$ sur V_n et l'application d'un théorème classique de E. Hopf sur les opérations de type elliptique permet d'en déduire soit que ϕ est à dérivée covariante nulle, soit que ϕ est nulle. Dans l'hypothèse où la courbure de Ricci est défini positive, on retrouve ainsi le théorème de Myers affirmant que le premier nombre de Betti est nul. Pour une V_n réalisable localement comme hypersurface de l'espace euclidien et localement convexe, on voit que si sa plus petite courbure principale est au moins la moitié de la plus grande les nombres de Betti de V_n sont ceux de la sphére. Un grand nombre de résultats sur les nombres de Betti peuvent être obtenus, sous des hypothéses variées sur la courbure, les résultats cités pouvant être considérés comme typiques.

La même méthode a été employé systématiquement par Bochner, puis par Yano pour l'étude des vecteurs de Killing et de leurs généralisations. Lors'que la courbure de Ricci est négative, la variété V_n n'admet pas de groupe à 1 paramètre d'isométries. Je signalerai aussi que tout espace homogène riemannien compact à courbure de Ricci nulle est nécessairement localement euclidien.

La rédaction du livre due à Yano est remarquablement claire et se suffit à elle-même. Un premier chapitre rappelle les éléments nécessaires de géométrie riemannienne. Les chapitres II et III sont consacrés aux formes harmoniques et aux vecteurs et tenseurs de Killing; on y trouvera une démonstration détaillée du théorème de Hopf, des conditions nécessaires et suffisantes commodes (dont l'une est due à de Rham) pour que sur une V_n compacte orientable un tenseur soit harmonique ou soit un tenseur de Killing. L'une des conséquences est que, sur une V_n compacte orientable, tout groupe à 1 paramètre de collinéations affines est un groupe d'isométries (Yano).

Les deux chapitres suivants sont relatifs d'une part aux variétés à courbure constante ou à courbure conforme ou projective nulle (au sens de Weyl), d'autre part aux variétés dont la courbure diffère suffisamment peu de celle des précédentes. Différentes définitions de "l'écart" de courbure sont données et leurs conséquences pour les nombres de Betti étudiées. Un chapitre est consacré aux variétés des groupes semi-simples compacts, un autre aux intéressants travaux de Bochner sur les variétés kählériennes; on voit par exemple que si une variété kählerienne admet une courbure de Ricci négative, il n'existe pas de vecteurs analytiques (contravariants) et par conséquent pas de groupe à 1 paramètre d'automorphismes analytiques, ce qui est en liaison avec des faits bien connus en géométrie algébrique. On trouvera aussi exposés les résultats classiques sur les nombres de Betti des variétés kählériennes.

Dans un chapitre spécial, une généralisation des résultats sur les formes harmoniques est tentée pour les variétés à connexion euclidienne (avec torsion). Les formes "pseudo harmoniques" (c'est-à-dire satisfaisant aux mêmes conditions que les formes harmoniques lorsqu'elles sont exprimées en termes de la connexion) n'ayant pas reçu d'interprétation topologique, l'intérêt des résultats est strictement du domaine géométrique.

Une note finale, due à Bochner, indique brièvement différents travaux et des voies de recherche: elle concerne les espaces symétriques (au sens d'Elie Cartan) et leurs généralisations, la convexité introduite par Lichnerowicz, les variétés minima, différents travaux sur les variétés analytiques complexes et leurs plongements, le théorème de Gauss-Bonnet ainsi que des études sur les variétés non compactes.

Ce compte rendu ne peut donner qu'une idée de la richesse de cet excellent livre qui doit avoir de nombreux lecteurs et susciter des travaux. *A. Lichnerowicz* (Paris).

Yano, Kentaro. The harmonic tensor and the Killing tensor. Sûgaku **4**, 205–215 (1953). (Japanese)

This paper was written, the reviewer thinks, as a brief introduction for young Japanese mathematicians to the theory initiated by S. Bochner [Bull. Amer. Math. Soc. **52**, 776–797 (1946); these Rev. **8**, 230] and followed by I. Mogi [Kōdai Math. Sem. Rep. **1950**, 61–66; these Rev. **12**, 746], A. Lichnerowicz [C. R. Acad. Sci. Paris **226**, 1678–1680 (1948); these Rev. **9**, 618], the author himself [Ann. of Math. (2) **55**, 38–45, 328–347 (1952); these Rev. **13**, 689; **14**, 88], and others. All the contents are included in Chaps. II, III, and VIII of the book reviewed above.

S. Sasaki (Sendai).

Haantjes, J., and Laman, G. On the definition of geometric objects. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **56** = Indagationes Math. **15**, 208–215, 216–222 (1953).

Les auteurs se proposent de définir la notion d'objet géométrique au moyen d'espaces fibrés convenables. Considérons un ensemble de n polynomes à n variables réelles, de degrés $\leqq s$, dépourvus de termes constants et pour lequels les coefficients des termes de degré 1 définissent une matrice régulière. Etant donnés deux tels ensembles, en substituant aux variables de l'un les polynomes de l'autre et omettant les termes de degré $> s$, on obtient un nouvel ensemble jouissant des mêmes propriétés et qui est dit leur produit. Ce produit définit une structure de groupe L_n^s et L_n^s peut être doué d'une manière naturelle d'une structure de groupe topologique. Cela posé, donnons nous une variété différentiable X_n^r, de dimension n, de classe r, un groupe G opérant effectivement sur un espace de Hausdorff Y et assumons que si U est un ouvert de G, Uy_0 ($y_0 \in Y$) est un ouvert dans Gy_0. Supposons que pour chaque $x \in X_n^r$ il existe un homomorphisme $h(x)$ de L_n^s ($s \leqq r$) sur G dépendant continûment de x. Avec X_n^r, G, Y et $h(x)$ on définit d'une manière naturelle un espace fibré E, l'espace fibré de classe $\leqq s$ des objets géométriques de type $h(x)$ [Ehresmann, dans une terminologie différente a considéré les mêmes espaces fibrés]. Une section de E sera un champ d'objets géométriques du type considéré. La dimension de Y est appelée la dimension de l'objet géométrique.

Deux tels espaces E et E', de même base X_n^r, fibres Y et Y', sont dits équivalents par les auteurs s'il existe un homéomorphisme $\lambda(n)$ de Y sur Y' dépendant continûment de x et tel que

$$g' = \lambda(x) g \lambda^{-1}(x) \quad [g = h(x)A,\ g' = h'(x)A,\ A \in L_n^s].$$

Cette définition diffère de l'équivalence usuelle pour les espaces fibrés. Un espace E d'objets géométriques est transitif si G opère transitivement sur Y; E est alors équivalent à un espace de fibre L_n^s/K et groupe $L_n^s/\bar{K}$, où K est sous-groupe fermé de L_n^s et $\bar{K}$ le plus grand sous-groupe de K invariant dans L_n^s. Inversement à tout sous-groupe fermé K de L_n^s correspond un objet géométrique; pour que les objets soient équivalents, il faut et il suffit que les groupes K correspondants soient conjugués les uns des autres.

Dans la seconde partie, les auteurs déterminent les objets géométriques transitifs de dimension 1 et classe 1 au sens précédent. La composante K_0 de l'identité de K est en général le groupe linéaire réel unimodulaire. Pour $n=2$, il s'y ajoute le groupe ensemble des éléments A de L_n^1 tels que $a_1^1>0$, $a_2^2>0$, $a_1^2=0$. On en déduit aisément une liste exhaustive des objets géométriques cherchés non équivalents [au sujet, voir aussi S. Gołąb, Ann. Soc. Polon. Math. **23**, 79–89 (1950) [ces Rev. **12**, 749] où la définition des objets géométriques est plus stricte]. *A. Lichnerowicz.*

Schouten, J. A. On the differential operators of first order in tensor calculus. Math. Centrum Amsterdam. Rapport **ZW 1953-012**, 6 pp. (1953).

The author reviews first the idea of a differential operator as differential concomitant: 1) The covariant derivative of a tensor T in a Riemann space is clearly a differential concomitant of T and the corresponding metric tensor $g_{\lambda\mu}$. 2) The Lie-Ślebodziński derivative of an object Ω is a differential concomitant of Ω and the underlying vector. This derivative has been generalized by the author [Nederl. Akad. Wetensch., Proc. **43**, 449–452 (1940); these Rev. **2**, 200]. 3) Investigating the conditions that all pairs of eigendirections of a tensor field $h_\lambda{}^\nu$ be X_2-building, Nijenhuis found a simple differential concomitant of two tensors $h^a{}_\lambda{}^\nu$ $(a=1, 2)$, namely (there is an obvious misprint in the equation 10) of the paper)

$$(\partial_\alpha h^{(1\ 2)}_{[\lambda}{}^\nu)h^{(1\ 2)}_{\mu]}{}^\alpha - h^{(1\ 2)}_\alpha{}^\nu \partial_{[\mu} h^{(1\ 2)}_{\lambda]}{}^\alpha.$$

4) Finally the author shows how in some simple cases new differential concomitants might be found. Example: The negative Lie-Ślebodziński derivative $-\underset{v}{\mathfrak{L}} h_\lambda{}^\nu$ of a tensor $h_\lambda{}^\nu$ may be looked upon as a kind of differential operation $O_\lambda v^\nu$ applied to v^ν

$$O_\lambda v^\nu = -\underset{v}{\mathfrak{L}} h_\lambda{}^\nu.$$

Then $v^\nu O_\lambda w^\mu + w^\mu O_\lambda v^\nu$ may be written in a form containing not only $u^{\nu\lambda} = v^\nu w^\lambda$ but also some additional terms. Substituting for these latter other terms, one obtains under certain algebraic conditions for $u^{\nu\mu}$ and $h_\lambda{}^\nu$ a new differential concomitant of these two tensors. Two other examples of the same method are given. *V. Hlavatý.*

***Inzinger, Rudolf. Zur Differentialgeometrie der Faltungsgruppe im Hilbertschen Raum.** Convegno Internazionale di Geometria Differenziale, Italia, 1953, pp. 300–301. Edizioni Cremonese, Roma, 1954. 4000 Lire.

NUMERICAL AND GRAPHICAL METHODS

Lotkin, M., and Young, M. E. Table of binomial coefficients. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Memorandum Rep. no. **652**, 33 pp. (1953).

This table gives values to twenty significant figures of

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

for integers n, r such that $0 \leqq r \leqq [\frac{1}{2}(n+1)]$, and $1 \leqq n \leqq 100$. It was derived from exact values computed by Pascal's recurrence

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

This is the most extensive table of the numbers ever published. A previous table of Peters and Stein [Peters, Zehnstellige Logarithmentafel, Bd. I, Berlin, 1922, Anhang, pp. 69–82] is for $n \leqq 60$. *D. H. Lehmer.*

Slater, L. J. The evaluation of the basic confluent hypergeometric functions. Proc. Cambridge Philos. Soc. **50**, 404–413 (1954).

The basic analogues

$$(1) \qquad \sum_{n=0}^{\infty} \frac{(q^a)_n y^n}{(q^b)_n (q)_n}$$

$$(2) \qquad \sum_{n=0}^{\infty} \frac{(q^a)_n y^n q^{n(n+1)/2}}{(q^b)_n (q)_n}$$

of confluent hypergeometric functions, where

$$(q^a)_0 = 1, \quad (q^a)_n = (1-q^a)(1-q^{a+1}) \cdots (1-q^{a+n-1}), \quad n = 1, 2, \cdots,$$

are discussed in this paper. The paper contains all formulas (such as recurrence relations, asymptotic expansions) which are necessary for the numerical evaluation of these functions, and also numerical tables of (1) for $a=0(.2)2$, $b=.2(.2)1$, $q=.9$, $y=.1(.1).9$ and of (2) for $a=0.(.2)2$, $b=.2(.2)1$, $q=.9$ and $y=.1(.1)1$, and $q=.99$, $y=.01(.01).1$. The tables were computed on the EDSAC in the Cambridge University Mathematical Laboratory. *A. Erdélyi.*

Guest, P. G. Grouping methods in the fitting of polynomials to equally spaced observations. Biometrika **41**, 62–76 (1954).

Given n equally spaced observations. The first part presents methods of fitting by use of N groups $(n=rN)$, including estimated standard errors. Relative efficiencies are computed, including the case when some observations must be omitted because n is prime; the loss of efficiency in the latter case may be very serious, especially for high-degree coefficients. A second part is devoted to the use of step-functions. By use of double-step functions, high efficiency is obtained for second and third degree operations. The chief weakness of step-function fitting is the difficulty of estimating standard errors and the degree of polynomial to be used; it is particularly useful when previous experience indicates a linear relationship is satisfactory. Comparative time studies are included of the various methods of fitting. *R. L. Anderson.*

Young, Andrew. Techniques for the summation of products on Hollerith and National accounting machines. Quart. J. Mech. Appl. Math. **7**, 136–151 (1954).

Freytag, H. Rechnerischer Ausgleich von Messkurvenscharen. Ing.-Arch. **22**, 194–202 (1954).

Schmetterer, Leopold. Ein Beitrag zur mechanischen Ausgleichung. Statist. Vierteljschr. **7**, 90–96 (1954).

Gray, H. J., Jr. Numerical methods in digital real-time simulation. Quart. Appl. Math. **12**, 133–140 (1954).

En vue de leur résolution par un simulateur l'auteur étudie les systèmes linéaires à coefficients constants. En raison de la faible précision desirée, il se borne à comparer les racines caractéristiques à plus grande partie réelle de

la solution exacte et de la solution approchée. Les résultats sont donnés sous la forme d'un plan complexe où l'on lit l'une des racines en fonction de l'autre. La methode s'applique à des formules de résolution approchée très générales, mais la représentation dans le plan complexe n'est donnée explicitement que pour une formule bien déterminée.

J. Kuntzmann (Grenoble).

Abramowitz, Milton. On the practical evaluation of integrals. J. Soc. Indust. Appl. Math. **2**, 20–35 (1954).

The paper is dedicated to some rules of practice which are explained by typical examples. Superfluous parameters are eliminated, and if possible, the integral is reduced to a known form. In other cases an evaluation may be carried out by limiting procedures such as are used in some integrals involving Bessel functions. It is also pointed out that functional relationships, termwise integration, reduction to a differential equation, extraction of a singular part, the Laplace transform and the use of contour integrals may help in the evaluation problem. *H. Bückner.*

Luke, Yudell L. On the computation of oscillatory integrals. Proc. Cambridge Philos. Soc. **50**, 269–277 (1954).

The author considers the integral:

$$\int_{-nrh}^{nrh} f(x) \exp(i\lambda x)dx$$

and assumes that $f(x)$ is tabulated at $(2n+1)$ points such that the interval between successive points is of constant width h. The factor r is introduced to permit integration over a range which differs from that for which $f(x)$ is tabulated. By using either Stirling's or Lagrange's method, the function $f(x)$ can be represented by a polynomial, and both cases are studied in detail. A number of tables giving the coefficients, needed in the computation are given. Also the error term is treated rather extensively and corresponding pictorial representations are given. *H. A. Lauwerier.*

***Edman, J. L. Graphical solution of simultaneous second order non-linear differential equations.** Proceedings of The First Midwestern Conference on Solid Mechanics, April, 1953, pp. 170–174. The Engineering Experiment Station, University of Illinois, Urbana, Ill., 1954.

The author reduces the second-order equations

$$\ddot{x}_i = F_i(\dot{x}_j, x_k, t)$$

to first-order equations in the dependent variables by the substitutions $\dot{x}_i = y_i$, $\ddot{x}_i = y_i dy_i/dx_i$. He then proceeds to obtain a graphical solution in the form of a polygon, regaining his independent variable t from the relation $\delta t = \delta x_i/y_i$. The method is essentially similar to that attributed by Kamke [Differentialgleichungen. Lösungsmethoden und Lösungen, Bd. I, 3. Aufl., Akademische Verlagsgesellschaft, Leipzig, 1944, p. 178; these Rev. **9**, 33] to Braun. *J. G. L. Michel* (Teddington).

***Young, Dana. Forced vibration of system with non-linear restoring force.** Proceedings of The First Midwestern Conference on Solid Mechanics, April, 1953, pp. 164–169. The Engineering Experiment Station, University of Illinois, Urbana, Ill., 1954.

This paper is concerned with the study of the behaviour of solutions to the equation [Duffing's equation, cf. Duffing, Erzwungene Schwingungen bei veränderlicher Eigenfrequenz . . . , Vieweg, Braunschweig, 1918]

$$\ddot{x}+\alpha\dot{x}+x+\beta x^3=\sin\omega t. \tag{1}$$

The amplitude and phase of stable harmonic solutions were obtained on REAC for a number of values of the parameters, and compared with analytical approximations based on the assumption that the true solution can be represented by the first two terms of an odd order Fourier series. The comparison was reasonably good, but the relatively crude REAC technique allowed only stable harmonic solutions to be obtained. The regions of unstable harmonic solution were investigated by considering the first-order variational equation for a variation in initial conditions; for an undamped equation with a modified forcing term the variational equation admits of analytical solutions in terms of Mathieu functions. The region of unstable harmonic solutions to equation (1) was shown to coincide with the major region of unstable solutions to Mathieu's equation. Peculiarities in the behaviour of the solution in neighbourhoods corresponding to other regions of instability of the Mathieu equation were noted. The authors are studying further unanswered questions raised by this investigation.

J. G. L. Michel (Teddington).

Schwarz, Eleonore. Numerische Lösung des Randwertproblems der Potentialgleichung mit Hilfe von Lochkarten. Z. Angew. Math. Mech. **34**, 237–240 (1954).

The numerical solution of the boundary value problem for the potential equation $\partial^2u/\partial x^2+\partial^2u/\partial y^2=0$ can be determined by solving the system of linear equations given by the finite difference formula

$$u_{i,k}=.20(u_{i-1,k}+u_{i+1,k}+u_{i,k-1}+u_{i,k+1}) +.05(u_{i-1,k-1}+u_{i+1,k-1}+u_{i-1,k+1}+u_{i+1,k+1}),$$

for each interior net point $P_{i,k}$ of a square lattice. The author describes in detail an efficient method for performing with punch-card equipment the simple iterative solution of the above equations (based on eight neighboring points).

E. Isaacson (New York, N. Y.).

Stein, Marvin L. On methods for obtaining solutions of fixed end-point problems in the calculus of variations. J. Research Nat. Bur. Standards **50**, 277–297 (1953).

The author considers two iterative methods for obtaining solutions of the Euler equations of an integral

$$I(y)=\int_a^b f(x, y_1(x), \cdots, y_n(x), y_1'(x), \cdots, y_n'(x))dx$$

satisfying given boundary conditions. The first method is an extension of Newton's method and is applicable to a more general system

$$T_i(x, y)=0, \quad y_i(a)=\rho_i, \quad y_i(b)=\sigma_i,$$

where T_i is an integral form of differential equations

$$\frac{d}{dx}g_i(x, y, y')=h_i(x, y, y'),$$

which vanishes at $x=a$ and $x=b$. When $g_i=f_{y_i'}$, $h_i=f_{y_i}$, the function $T_i(x, y)$ can be interpreted as the gradient of I along the curve y. Convergence theorems are given for this method. It is shown in particular that the convergence is quadratic in character. It is shown further that the method is applicable even if the solution sought has on it conjugate points of its initial point. However, the end points must not be conjugate to each other. The second method is based on the iteration

$$y_{i1}=y_{i0}+\alpha T_i(x, y_0),$$

where α is a constant. It is a gradient method and converges

to a solution under more restrictive conditions than does Newton's method. However it is simpler to perform. Numerical illustrations are given. *M. R. Hestenes.*

***Bogdanoff, J. L., Goldberg, J. E., and Lo, Hsu. Application of Volterra linear integral equations to the numerical solution of beam vibration problems. I.** Proceedings of The First Midwestern Conference on Solid Mechanics, April, 1953, pp. 81–88. The Engineering Experiment Station, University of Illinois, Urbana, Ill., 1954.

Most beam vibration problems are related to one space coordinate x. They lead to eigenvalue problems of the type $L(\lambda)w(x)=0$, $0\leqq x\leqq 1$, with $w(x)$ describing the amplitude, λ the eigenvalue and $L(\lambda)$ a linear differential operator applied to w. Values of w and/or of its derivatives are prescribed at $x=0$ and $x=1$. The authors are especially concerned with the case when the beam does not have a uniform mass distribution and a uniform stiffness. To handle the problem, they set up a Volterra integral equation $w(x)=\int_0^x K(x,s,\lambda)w(s)ds+F$ with $F=\sum_{k=1}^n C_k f_k(x,\lambda)$, where the C_k are integration constants. The solutions of this integral equation satisfy the boundary conditions for $x=0$ for each set of integration constants. With an estimate of λ, the solution of the integral equation is evaluated for the special cases $F=f_k$, and the general solution is obtained by linear combination of these partial solutions. The boundary conditions for $x=1$ lead to a system of homogeneous simultaneous equations for the C_k. Then λ is varied by trial and error so as to make the determinant of this system vanish.

For the numerical evaluation of the partial solutions of the integral equation Huber's method is used. According to Huber, the interval $\langle 0, 1\rangle$ is divided into m equal steps, and the integral equation is taken into account for the pivotal points; w is approximated by piecewise linear functions, each linear part belonging to one of the intervals between adjacent pivotal points. This method leads to a system of linear equations with a triangular matrix for the pivotal values of w. Some numerical examples are presented, which refer to uniform beams, in order to make comparisons with exact solutions possible. *H. Bückner.*

Bogdanoff, J. L., Goldberg, J. E., and Lo, Hsu. Application of Volterra linear integral equations to the numerical solution of vibration problems. II. J. Aeronaut. Sci. **21**, 383–388, 403 (1954).

The method developed in the paper reviewed above is applied to nonuniform beams, and numerical examples refer to the torsional vibration of a fixed-free beam of variable section, to the torsional vibration of a fixed-free beam carrying a rigid disc and to the lateral vibration of a nonuniform cantilever beam. The main features of the method are discussed in more detail. It is pointed out that the Huber method, which is used to solve a Volterra integral equation, does not allow one to deal with more than the first m eigenvalues, when m is the number of subdivisions of the basic interval according to Huber. The authors suggest improving the Huber method by using parabolic rather than linear approximations. This however has already been done by C. Wagner [J. Math. Physics **32**, 289–301 (1954); these Rev. **15**, 651]. *H. Bückner* (Schenectady, N. Y.).

Ham, J. M. A computer for solving integral formulations of engineering problems by methods of successive approximations. Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Mass., Tech. Rep. No. **241**, ii+54 pp. (1953).

***Orsini, Luiz de Queiroz, and Camargo Vieira, Rui. An electrolytic tank for the study of circuits and fields.** New research techniques in physics, pp. 389–394. Symposium organized by the Academia Brasileira de Ciências and Centro de Cooperación Científica para America Latina (UNESCO) under the auspices of the Conselho Nacional de Pesquisas do Brasil, Rio de Janeiro and São Paulo, July 15–29, 1952. Rio de Janeiro, 1954.

Castagno, Aldo. Sulle trasformazioni conformi eseguite sperimentalmente con la vasca elettrica. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **87**, 310–325 (1953).

***Flater, Erich, and Franz, Kurt. Electronic multiplier based upon impulses modulated in length and in amplitude.** New research techniques in physics, pp. 379–385. Symposium organized by the Academia Brasileira de Ciências and Centro de Cooperación Científica para America Latina (UNESCO) under the auspices of the Conselho Nacional de Pesquisas do Brasil, Rio de Janeiro and São Paulo, July 15–29, 1952. Rio de Janeiro, 1954. (Spanish)

Curtis, G. C. Nomograms for the solution of the sound-ranging problem in a plane. Quart. J. Mech. Appl. Math. **7**, 129–135 (1954).

Nomograms are given for the determination of the plane coordinates x and y of a sound source from the differences in times of arrival of the sound at three collinear receivers in the plane. *P. W. Ketchum* (Urbana, Ill.).

Hazay, I. Mathematische Grundlage zur einheitlichen Tabelle verschieden angeordneter, winkeltreuer Zylinderprojectionen. Acta Tech. Acad. Sci. Hungar. **8**, 369–388 (1954). (Russian, French and English summaries)

The problem of transformation of geographical data given with reference to an oblique conformal cylindrical projection with an arbitrary base point is considered by means of a correlation between the projection coordinates and the geographical coordinates of the sphere. One set is expressed as a rapidly converging power series in the other. Selecting one coordinate as the series parameter, the coefficients are developed for convenient tabulation. Numerical examples illustrate the application of the method to cartographic problems in Hungary. *N. A. Hall* (Minneapolis, Minn.).

Cazenave, René. Méthode graphique de résolution des triangles sphériques. Cahier de Physique no. **48**, 67–71 (1954).

ASTRONOMY

Lindblad, Bertil, and Nahon, Fernand. On the theory of star-streaming. Stockholms Observatoriums Annaler **18**, no. 2, 22 pp. (1954).

Nearly circular orbits in the (x, y)-plane in a gravitational field which has symmetry about this plane and has also cylindrical symmetry about the z-axis when described in a frame of reference rotating with an angular velocity ω_z are epicycles with a determinate characteristic frequency. The equations governing these epicyclic orbits can be explicitly written down [cf. S. Chandrasekhar, Principles of stellar

dynamics, Univ. of Chicago Press, 1942, pp. 156–158; these Rev. 4, 57] and involve four constants of integration c, c_1, t_0 and t_1 (say). According to Liouville's theorem an arbitrary function of the integrals of the equations of motion which are these constants represent possible solutions for the distribution function in the four-dimensional phase space. In this paper the authors consider a distribution function of the form e^{-Q} where Q is a polynomial in the integrals which includes terms which are at most quadratic in the velocities w and v in the orbits relative to the circular orbit at which the particle finds itself instantaneously. The object of considering such a distribution function is that the authors believe that certain aspects of galactic evolution can be understood in terms of it. *S. Chandrasekhar.*

Ževakin, S. A. Discrete model of a star. II. Akad. Nauk SSSR. Astr. Žurnal **30**, 52–63 (1953). (Russian)

[For part I see same Žurnal **29**, 37–48 (1952); these Rev. **14**, 212.] The model treated has the mass distributed only on the surfaces of two concentric spheres, the "nucleus" and the "envelope", each filled with elastic weightless gas. The author considers the first-order terms in the equations of motion of this stellar model with a view towards application to variable stars, particularly the cepheids.

R. G. Langebartel (Urbana, Ill.).

Horak, Henry G. The transfer of radiation by an emitting atmosphere. III. Astrophys. J. **119**, 640–646 (1954).

The present paper continues the writer's earlier investigations of the radiative transfer by an emitting plane-parallel atmosphere [same J. **116**, 477–490 (1952); these Rev. **14**, 804] with a uniform distribution of energy sources, and considers two cases: (a) transfer by scattering in accordance with Rayleigh phase function $\frac{3}{4}(1+\cos^2\theta)$; (b) Rayleigh scattering in which the polarization of scattered light is taken into account (the light emitted by the atmosphere being supposed to be nonpolarized). Both problems are attacked by the "method of discrete ordinates"; in the latter case, the solution of the homogeneous problem was already given by Chandrasekhar [ibid. **106**, 152–216 (1947), p. 184; these Rev. **9**, 444], so that only the requisite particular integrals had to be constructed. The emergent radiation field has been calculated for problem (b) in a finite Rayleigh-scattering atmosphere of optical thickness 0.20, and the results are given in the form of a table. *Z. Kopal.*

de Orús, Juan J. Contribution to Chandrasekhar's theory of the dynamics of stellar systems. Collectanea Math. **5**, 121–149 (1 plate) (1952). (Spanish. English summary)

The circumstances under which Liouville's equation

$$U\frac{\partial\Psi}{\partial x}+V\frac{\partial\Psi}{\partial y}+W\frac{\partial\Psi}{\partial z}-\frac{\partial V}{\partial x}\frac{\partial\Psi}{\partial U}-\frac{\partial V}{\partial y}\frac{\partial\Psi}{\partial V}-\frac{\partial V}{\partial z}\frac{\partial\Psi}{\partial W}=0,$$

relating the distribution function $\Psi(x, y, z, U, V, W)$ and the gravitational potential $V(x, y, z)$ allows a solution for Ψ which is a quadratic function of the velocities U, V, W of the form $Q+\sigma$ where

$$(*)\quad Q=a(U-U_0)^2+b(V-V_0)^2+c(W-W_0)^2+2f(V-V_0)(W-W_0)+2g(W-W_0)(U-U_0)+2h(U-U_0)(V-V_0),$$

where $a, b, c, f, g, h, U_0, V_0, W_0$ and σ are functions of x, y, z, has been investigated in great detail. The present paper is a contribution to the theory as developed by the reviewer [Principles of stellar dynamics, Univ. of Chicago Press, 1942, Chapter 4; these Rev. **4**, 57].

The paper is in three parts: In the first part the basic equations of the theory are written in tensor notation which makes for compactness. In the second part some additional theorems are proved. In particular, it is shown that an equation of continuity in ordinary space follows in which the velocity of the local centroid U_0, V_0, W_0 plays the role of the ordinary velocity. The main part of the paper is however devoted to the following problem: Considering the case when the solutions for a, b, c, f, g, h given in "Principles of stellar dynamics" (pp. 96–97) are symmetrical about the origin, the author asks the conditions under which the determinant $|A|$ of the quadratic form (*) will be a function of (x^2+y^2) and z^2 only; and shows that this will be the case when

$$a=k_1+k_2y^2+k_4z^2;\quad b=k_1+k_2x^2+k_4z^2;\quad c=k_3+k_4(x^2+y^2),\quad f=-k_4yz,\quad g=-k_4xz,\quad h=-k_2xy$$

and

$$|A|=[k_2(x^2+y^2)+k_4z^2+k_1][k_1k_4(x^2+y^2)+k_2k_4z^2+k_1k_3],$$

where k_1, k_2, k_3 and k_4 are constants. In terms of the foregoing solution the author seeks to explain the observed forms of elliptical nebulae. *S. Chandrasekhar.*

Limber, D. Nelson. Analysis of counts of the extragalactic nebulae in terms of a fluctuating density field. II. Astrophys. J. **119**, 655–681 (1954).

In a previous paper [same J. **117**, 134–144 (1953); these Rev. **14**, 803] the author developed a method for analyzing the counts of the extragalactic nebulae in terms of a fluctuating density-field of the nebulae in space, based on an earlier work by Chandrasekhar and Münch [ibid. **115**, 103–123 (1952); these Rev. **13**, 786]. In the present paper this theory has been applied to an analysis of the Lick nebular counts (as yet unpublished) by Shane and Wirtanen, in accordance with a model which assumes that the universe of the nebulae is static in time and homogeneous with respect to the absolute magnitudes and the spatial-distribution properties of the nebulae within it.

A comparison of the theory with the observations indicates that: (1) the effects of fluctuations in the absorption within our own galaxy can be neglected to a first approximation; (2) a dispersion of approximately 2 magnitudes in absolute magnitudes of the nebulae is found to be in best accord with the observations; (3) the mean square of the deviations in space density of the nebulae from the mean is about 25 times the square of the mean density. From a straightforward analysis of the counts the optical half-thickness of the galaxy appears to be approximately 0.76, and the density of extragalactic nebulae approximately 6×10^{-19} nebulae per cubic parsec. *Z. Kopal.*

RELATIVITY

Einstein, A., and Kaufman, B. Algebraic properties of the field in the relativistic theory of the asymmetric field. Ann. of Math. (2) **59**, 230–244 (1954).

In the theory of the non-symmetric field (g_{α} a non-symmetric tensor field) it is necessary for the signature of the field to be constant throughout space in order that a distinction between time-like and space-like directions should exist at each point of the continuum. The condition for this is that the determinant of the symmetric part of g_{α} should not vanish at any point, and it is shown that

this condition is a consequence of two postulates: (1) that $\det(g_{ab})\neq 0$, and (2) that the Γ field should be 'regular', i.e. that $\Gamma^{\lambda}_{\mu a}$ should exist and be determined uniquely by the equation $g_{ab;\,c}=0$. It is shown that postulate (2) is not a consequence of postulate (1) together with the constancy of signature, and the paper concludes with a detailed solution of the algebraic equations for Γ. *A. G. Walker.*

Klein, O. On a class of spherically symmetric solutions of Einstein's gravitational equations. Ark. Fys. **7**, 487–496 (1954).

In the theory of polytropic equilibria the pressure p is related to the density ρ by a relationship of the form $p=k\rho^{(n+1)/n}$. By assuming a relationship between the pressure and gravitational potentials the author attempts to provide a relativistic generalization of this theory.

M. Wyman (Edmonton, Alta.).

Hatalkar, M. H. Theory of elementary particles in general relativity. Physical Rev. (2) **94**, 1472–1475 (1954).

Pauli [Ann. Physik (5) **18**, 337–372 (1933)] generalized Dirac's equation for spin-$\frac{1}{2}$ particles to general coordinates by introducing a spin-affine transformation. The author carries this program through for Bhabha's equation for elementary particles [Rev. Modern Physics **17**, 200–216 (1945); these Rev. **7**, 272]. This includes, as special cases, equations for spin-zero and spin-one particles.

A. Salam (Cambridge, England).

Schöpf, H.-G. Ableitung der relativistischen Mechanik aus der Einsteinschen Massen-Energie-Formel. Wissensch. Z. Univ. Greifswald. Math.-Nat. Reihe **3**, 183–186 (1954).

Bonnor, W. B. Static magnetic fields in general relativity. Proc. Phys. Soc. Sect. A. **67**, 225–232 (1954).

By proving that the dual of the electromagnetic field tensor also satisfies the electromagnetic field equations of general relativity the author derives a method for converting a class of electrostatic solutions of the field equations into magnetostatic solutions. Several examples illustrating the method are given. *M. Wyman* (Edmonton, Alberta).

Jaiswal, J. P. A note on electromagnetic phenomena in gravitational fields. Ganita **2**, 62–64 (1951).

L'auteur reprend certains travaux de E. T. Whittaker relatifs au potentiel vecteur électromagnétique solution (à l'extérieur de la matière) des équations relativistes de l'électromagnétisme par des champs simples [Proc. Roy. Soc. London. Ser. A. **116**, 720–735 (1927); **120**, 1–13 (1928)]. Les cas de la métrique de la relativité restreinte et du champ quasi uniforme

$$ds^2=(1+2gc^{-2}x)dt^2-c^{-2}[(1+2gc^{-2}x)^{-1}dx^2+dy^2+dz^2]$$

sont traités. L'auteur montre que les valeurs indiquées par Whittaker sont le double des valeurs correctes et indique l'origine de cette erreur. *A. Lichnerowicz* (Paris).

Mariot, Louis. Le champ électromagnétique singulier. C. R. Acad. Sci. Paris **238**, 2055–2056 (1954).

If, in general relativity, the electromagnetic six-vector $F_{\alpha\beta}$ satisfies the equations $F\wedge F=0$, $F\wedge *F=0$, then, in a local orthonormal frame, the electric and magnetic 3-vectors are perpendicular. This case the author calls singular. If $\tau_{\alpha\beta}$ is the electromagnetic energy-tensor, then in the singular case there exists a null (i.e. isotropic) vector l_α such that $\tau_{\alpha\beta}=P^2 l_\alpha l_\beta$, where P^2 is a scalar [Ruse, Proc. London Math. Soc. (2) **41**, 302–322 (1936), p. 321]. Clearly l_α is an eigenvector of $\tau_{\alpha\beta}$, and is the only one that is null. Supposing that $F_{\alpha\beta}$ satisfies Maxwell's equations in vacuo, the author proves that the trajectories of the isotropic eigendirections thus defined are null geodesics. He concludes with the remark that a singular field can be interpreted as a flow of photons, and draws attention to the similarity of one of the equations for l^α (deduced from Maxwell's equations) to the equation of continuity for a perfect fluid. *H. S. Ruse.*

Mavridès, Stamatia. Sur le choix de la métrique et du champ électromagnétique en théorie unitaire d'Einstein. C. R. Acad. Sci. Paris **238**, 1566–1568 (1954).

Mavridès, Stamatia. Courant et charge en théorie unitaire d'Einstein. C. R. Acad. Sci. Paris **238**, 1643–1644 (1954).

Starting with Papapetrou's solution the author gives a discussion of a suitable interpretation of the field quantities representing the metric and electromagnetic field in Einstein's unified field theory. The results obtained by means of this interpretation are then compared with the theory of Born and Infeld. *M. Wyman* (Edmonton, Alta).

MECHANICS

Wunderlich, W. Ein merkwürdiges Zwölfstabgetriebe. Österreich. Ing.-Arch. **8**, 224–228 (1954).

If a 12-bar linkage has the form of a parallel projection of a cube onto a plane, it is composed of six parallelograms and it has two degrees of freedom. This linkage can be passed through transition stages to assume a position in which two, three, four or six parallelograms are replaced by crossed parallelograms (antiparallelograms). In the cases of three or four crossed parallelograms, there is only one degree of freedom. *M. Goldberg* (Washington, D. C.).

Duncan, W. J. A kinematic property of the articulated quadrilateral. Quart. J. Mech. Appl. Math. **7**, 222–225 (1954).

By an application of the theorem of Pappus, it is shown that if a movable quadrilateral linkage is moved so that the normals to the paths of two opposite vertices meet on the diagonal through the other pair of opposite vertices, then the normals to the paths of the latter vertices meet on the diagonal through the first pair of vertices. This property applies to spherical quadrilaterals also. *M. Goldberg.*

Rosenauer, N. Acceleration centre curves. Australian J. Appl. Sci. **5**, 103–115 (1954).

The acceleration a of any point A in a moving plane is expressed by $a=A\Gamma(\omega^4+\tau^2)^{1/2}$, where A is the distance of the point A from the acceleration center Γ, ω is the angular velocity of the moving plane and τ is the angular acceleration of the moving plane. The acceleration center is determined graphically by means of Joukowski's construction from the known accelerations of two points. The foregoing is used for the graphical construction of the paths, in the fixed and moving planes, of the acceleration center of a connecting rod of a crank mechanism. Analytic expressions

are derived from this graphical procedure to give more accurate results for remote intersections or small-angle intersections. *M. Goldberg* (Washington, D. C.).

Myasnikov, P. V. On a new method of singling out integrable cases of motion from the general problem of rotation of a heavy rigid body about a fixed point; a new particular case of motion. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk **9**, no. 3, 47–55 (1954). (Russian)

The author proposes to determine gyrations in which the mass center G is coplanar with the angular velocity and the angular momentum. This yields a third algebraic integral, $Aap+Bbq+Ccr=m$, a constant (a, b, c are the coordinates of G along the principal axes). The paper limits itself to the case when G is in a principal plane, say 0yz. Then, if $p\neq 0$, six old cases are obtained. To find a new one, $p=0$ is assumed, which obviously satisfies the conditions, and leaves us with the equations (α, β, γ are cosines of the vertical):

$$(1)\qquad (B-C)qr+Mg(b\gamma-c\beta)=0,\quad B\dot{q}=Mgc\alpha,$$
$$C\dot{r}=-Mgb\alpha;\quad \dot{\beta}=-r\alpha,\quad \gamma=q\alpha,$$

which seem simple enough. In the paper, however, they get some advanced treatment whose description will not be attempted here except for the remark that somehow it results in $m=0$, $B=C$, and some impressive formulas for the Euler angles. An elementary treatment would yield, through elimination of α, constant values for $2Mgb\beta-Cr^2$ and $2Mgc\gamma-Bq^2$. This, unless $m=B-C=0$, combined with the first of eqs. (1), yields constant values for q and r, hence $\alpha=0$, and, consequently, constant values for β and γ. Since $\dot{\alpha}=\beta r-\gamma q$, it follows that $q/r=\beta/\gamma$: the motion is a (Staude) permanent rotation about the vertical. If, however, $m=B-C=0$, we can assume $b=0$ and get, successively, $b\gamma-c\beta=0$, $\beta=0$, $r=0$, that is, a pendular motion about the horizontal axis 0y. All these cases are old.

A. W. Wundheiler (Chicago, Ill.).

Grammel, R. Der selbsterregte unsymmetrische Kreisel. Ing.-Arch. **22**, 73–97 (1954).

Let Tu be the torque, Ωw the angular velocity, u and w being unit vectors. Let a be the unit vector of the principal axis with either the largest, or the smallest moment of inertia, and b the unit vector on the middle principal axis. Let a prime denote the derivative with respect to time in the coordinate frame of the principal axes. The paper concerns itself with gyrations with fixed velocity direction ($w'=0$), fixed torque direction in the body ($u'=0$), or both. The trivial cases $ua=1$, $ub=1$ are excluded. The case $u'=0$ may occur when jet propulsion is used: hence the "self-excitement" of the title.

Both $u'=0$ and $w'=0$ if, and only if, Ωt is constant (t is the time) which implies that $(T\Omega)'=0$ (secs. 8 and 9). If $wa=1$ or $wb=1$ while $T'=0$, the problem reduces to quadratures. For $wa=1$ and a sufficiently large initial Ω_0, Ω tends towards infinity and $wu\to 1$. For sufficiently small Ω_0, Ωw will be periodic relative to the body. In the intermediate cases the motion is asymptotic towards a steady (stable or unstable) motion. This classification generalizes that of the Euler-Poinsot case, the intermediate cases corresponding to the separating polhodes (sec. 5). For $wb=1$, Ω never tends to infinity, unless $W_x^2/W_z^2=A(A-B)/C(B-C)$ (w in Poinsot's separating planes), and u_x/u_z is negative (if it is positive, the angular velocity is periodic in the body). The quadratures simplify in this case (sec. 6). The problem can still be reduced to quadratures when w is in the separating planes even though u is not on the middle axis but $u_x^2/u_z^2=A(B-C)/C(A-B)$. The author conjectures about the nature of the motions (sec. 7).

If $u'a=1$, $T'=0$, and $(w')^2-(w'a)^2$ has an initial value sufficiently smaller than T^2, successive approximations lead to motions of the same time as those in sec. 5 (sec. 3). If in these assumptions u and w are transposed, successive approximations are still possible (sec. 4). The approximations fail if one of the vectors is on the middle axis. But they are presumably useful if both vectors are close to a (sec. 5).

A. W. Wundheiler (Chicago, Ill.).

***Bouligand, Georges. Réflexions sur la mécanique classique.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 13–15. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

Wilkens, Alexander. Der Hodograph der Gravitation zweier Massenpunkte. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1953**, 149–161 (1954).

Der Hodograph der Geschwindigkeit bei der Planetenbewegung ist bekanntlich ein Kreis. Verf. untersucht den Hodograph der Beschleunigung; es zeigt sich dass damit $\mathfrak{r}$ gemeint ist, wenn r den Radius darstellt. Die Kurve wird zeicherisch und numerisch ausführlich diskutiert. (Es scheint Ref. dass es sich um eine rationale bizirkuläre Kurve 8. Ordnung handelt mit sechsfachem Punkt im Ursprung.)

O. Bottema (Delft).

Marsicano, Fénix Roberto. On the rigid body with three degrees of freedom. Atti Sem. Mat. Fis. Univ. Modena **6** (1951–52), 57–69 (1953). (Spanish)

The author considers a rigid body moving so that it remains in contact with a given surface and so that a point, fixed in the body, moves on a given curve. The fundamental kinematical and dynamical equations governing the motion are derived. In a special case, in which the body is a sphere, the surface a plane, and the curve a straight line, the equations are solved, and the motion is discussed in detail.

L. A. MacColl (New York, N. Y.).

Marsicano, Fénix Roberto. On the movement of a heavy nonhomogeneous sphere which moves without slipping on a fixed horizontal plane. Atti Sem. Mat. Fis. Univ. Modena **6** (1951–52), 70–75 (1953). (Spanish)

Assuming that the center of mass of the sphere coincides with the center of figure, the author derives the differential equations of motion for the system described in the title. Three first integrals are found, by means of which it is possible to determine the motion by quadratures. It is shown that under certain initial conditions the motion can be represented by the motion of an ellipsoid, fixed in the sphere, rolling on a circular cylinder fixed in space.

L. A. MacColl (New York, N. Y.).

Vacca, Maria Teresa. Soluzioni stazionarie nel moto di rotolamento di una sfera pesante non omogenea sopra un piano orizzontale. Atti Sem. Mat. Fis. Univ. Modena **6** (1951–52), 119–133 (1953).

It is assumed that the inhomogeneous sphere under consideration has its centroid at the center, and that it rolls without slipping on a horizontal plane. The paper gives an elaborate discussion of those motions for which the energy is stationary with respect to small variations of the motion (compatible with the non-holonomic constraint) which leave the angular momentum unaltered. No indications are given as to the physical significance of such motions.

L. A. MacColl (New York, N. Y.).

Zeuli, Tino. Moto di rotolamento di una sfera non omogenea pesante su un piano orizzontale. Atti Sem. Mat. Fis. Univ. Modena 6 (1951–52), 134–146 (1953).

The author studies the motions of a sphere, with a distinct center and centroid, which rolls without slipping on a horizontal plane. He investigates the stabilities of the motions in which the line joining the center and centroid is vertical, and the sphere rotates about this line. The results are interpreted as explaining the behavior of the so called tippe-top. *L. A. MacColl* (New York, N. Y.).

Minakov, A. P. On certain properties of the unforced steady motion of an ideally flexible inextensible cord (chain) in a fixed plane. Vestnik Moskov. Univ. Ser. Fiz.-Mat. Estest. Nauk 9, no. 3, 57–64 (1954). (Russian)

The first result of the paper is that a homogeneous cord described in the title and solicited by two forces of fixed directions a, b at two fixed points A and B of the plane retains its initial form and speed. Routh [Advanced rigid dynamics, 6th ed., Macmillan, London, 1930, art. 595] proves a much more general theorem of which a special case was proved in 1854 by Walton and Mackenzie [Solutions of the problems and riders . . . , Macmillan, London, 1854]: "If a uniform endless chain rests . . . subject to the action of forces depending only on the position . . . and to reactions of smooth surfaces, it will continue to move" retaining its initial form if the initial velocities are all tangential. Even this is much more general than the result of the paper. It is further shown that the resultant of the inertias can be expressed simply in terms of A, B, a, b, the length of the cord, its speed, and its linear density. The same is true for the virial about A or B, i.e., $\int(xF_x+yF_y)d\sigma$, F being the inertia of the element $d\sigma$.

A. W. Wundheiler (Chicago, Ill.).

Šurova, K. E. Remarks on the paper of K. E. Šurova, "Forming the variation of Poincaré's equations", Prikl. Mat. Meh. 17, no. 1, 1953. Akad. Nauk SSSR. Prikl. Mat. Meh. 18, 384 (1954). (Russian)

An error in the title paper [same journal 17, 123–124 (1953); these Rev. 14, 808] has been pointed out (some second derivatives of H have been omitted). Its correction adds the term $c_{\alpha\gamma}{}^{\sigma}\partial^{\beta\gamma}H\,p_\gamma\delta p_\beta$ to the right-hand side of the last equation of the review while the next-to-last one becomes an identity and must be replaced by the equation

$$d\delta q^\alpha/dt=c_{\gamma\beta}{}^{\alpha}\partial^{\beta}H\,\delta q^\gamma+X_\gamma\partial^{\alpha}H\,\delta q^\gamma+\partial^{\alpha\beta}H\,\delta p_\beta$$

derived by N. G. Četaev [ibid. 5, 253–262 (1941); these Rev. 4, 225]. *A. W. Wundheiler* (Chicago, Ill.).

Hydrodynamics, Aerodynamics, Acoustics

Coburn, N. Intrinsic relations satisfied by the vorticity and velocity vectors in fluid flow theory. Michigan Math. J. 1 (1952), 113–130 (1953).

Coburn, N. Note on my paper "Intrinsic relations satisfied by the vorticity and velocity vectors in fluid flow theory." Michigan Math. J. 2, 41–44 (1954).

These papers study general properties of the flow of a perfect fluid in terms of the principal directions of its stream field. For example, the author observes that the pressure does not vary in the direction of the binormal, and more generally he is able to state the quantities on which the variation of other flow quantities in the principal directions depends. *C. Truesdell* (Bloomington, Ind.).

***Garabedian, P. R. On free-surface flows.** Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 29–39. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00.

This is mainly a condensed version of a paper by Garabedian, Lewy, and Schiffer [Ann. of Math. (2) 56, 560–602 (1952); these Rev. 14, 810] on the existence and uniqueness of axially symmetric cavitation flow. *J. B. Serrin.*

***Stoker, J. J. Some remarks on radiation conditions.** Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 97–102. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00.

Dans un problème de propagation d'ondes dans un milieu s'étendant à l'infini, il est en général nécessaire, pour que la solution soit unique, de faire des hypothèses sur l'allure à l'infini de cette solution. Or il peut être difficile de fixer, a priori, ce comportement à l'infini, de telle manière que la solution ait les propriétés physiques voulues. Pour tourner cette difficulté on peut, en principe, obtenir le mouvement en question à partir de la solution d'un problème aux valeurs initiales, en faisant tendre le temps vers l'infini. C'est là le point de vue développé par l'auteur qui a réussi dans le cas particulier suivant à expliciter la solution du problème aux valeurs initiales ainsi posé et à effectuer le passage à la limite.

Le cas particulier envisagé concerne les petits mouvements périodiques d'un liquide pesant, engendrés par une pression appliquée en une région de la surface libre. Le mouvement est supposé plan, irrotationnel, en profondeur infinie, et régit par les équations linéarisées des petits mouvements. La pression perturbatrice étant de la forme $p(x, t)=\delta(x)e^{i\omega t}$ et le liquid initialement au repos, une transformation de Fourier sur le potentiel permet d'obtenir sous forme explicite la solution qui est unique. L'étude, lorsque le temps devient infini, montre que le mouvement tend alors vers un mouvement oscillatoire qui, au loin, donne lieu à une houle progressive, avec une longuer d'onde qui satisfait à la relation classique pour le cas d'une profondeur infinie.

R. Gerber (Toulon).

Stepanov, G. Yu. Construction of a grid with velocity distribution given on the circumference of a grid of circles. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 727–734 (1953). (Russian)

Relying on Chaplygin's approximation for subsonic gas flow, the author reduces the problem to the determination of the function which maps the plane of the profiles (the z-plane) on the plane of the grid of circles (the m-plane), the modulus of dz/dm being known on the boundary. A numerical illustration is worked out in detail.

L. M. Milne-Thomson (Greenwich).

***Reissner, Hans J. The irrotational flow pattern in an incompressible non-viscous fluid, produced by a circular double source (suction) disk-surface and by an inflow toward the suction surface perpendicular to the surface.** Mémoires sur la mécanique des fluides offerts à M. Dimitri P. Riabouchinsky, pp. 263–269. Publ. Sci. Tech. Ministère de l'Air, Paris, 1954. 3000 francs.

Lehrian, Doris E. A calculation of the complete downwash in three dimensions due to a rectangular vortex. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2771 (1949), 42 pp. (4 plates) (1953).

"A calculation of the complete downwash in three dimensions due to a rectangular vortex is given. The downwash is computed at selected positions, in planes normal to the plane of the vortex. Values are tabulated and a set of graphs is also included. Results are to be used in conjunction with same Rep. and Memoranda no. 2461 (1953) [these Rev. 14, 905].

"The complete downwash is calculated from the Glauert formula

$$F(x,y,z)=\left\{-\frac{x}{x^2+z^2}\left[\frac{y+1}{\{x^2+(y+1)^2+z^2\}^{1/2}}-\frac{y-1}{\{x^2+(y-1)^2+z^2\}^{1/2}}\right]-\frac{y-1}{(y-1)^2+z^2}\left[1-\frac{x}{\{x^2+(y-1)^2+z^2\}^{1/2}}\right]+\frac{y+1}{(y+1)^2+z^2}\left[1-\frac{x}{\{x^2+(y+1)^2+z^2\}^{1/2}}\right]\right\}."$$

E. Reissner (Cambridge, Mass.).

Muggia, Aldo. Sulla teoria delle superfici portanti. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. 87, 193–199 (1953).

Some line-vortex methods for the approximate calculation of the lift distribution on a three-dimensional wing are based on the assumption that the concentration of the bound vorticity at the quarter-chord yields the correct downwash at the three-quarter-chord. An analytical justification of this assumption is expounded in the present paper.

A. Robinson (Toronto, Ont.).

***Topakoğlu, Cavit. Sikiştirilamaz lüzuci bir akişkanin tor seklindeki bir boru içinde laminer ve daimi hareketi. [On the laminar steady motion of an incompressible viscous fluid in a torus-shaped tube.]** Thesis, Istanbul Technical University, Kutulmuş Basimevi, Istanbul, 1951. 30 pp.

The paper is concerned with the laminar steady motion of an incompressible viscous fluid in a curved pipe. It is assumed that the unknown functions (the velocity component in the circumferential direction and the velocity potential of the motion in the meridian plane) can be expanded in series of powers in a/R, where a is the radius of the pipe and $1/R$ is the curvature of the central line of the pipe. A system of nonhomogeneous equations is then obtained for the coefficients of powers of a/R by using the Navier-Stokes equations. Three terms of each series are found and the rate of flow due to a given pressure gradient is computed. The last result is compared with a previous analysis [W. R. Dean, Philos. Mag. (7) 4, 208–223 (1927)].

E. T. Onat (Ankara).

Morgan, G. W., and Kiely, J. P. Wave propagation in a viscous liquid contained in a flexible tube. J. Acoust. Soc. Amer. 26, 323–328 (1954).

The theory is linearized by assuming the waves small, the tube walls thin, and the wavelength large compared with the tube radius. Further simplifications are made based on the order of magnitudes of terms in the linearized expressions. A characteristic equation for the complex propagation constant is obtained and solved when the viscosity of the fluid is small on the one hand, or when either the viscosity is large or the frequency is low on the other hand. The theory is modified to take into consideration internal damping in the tube itself. *E. Pinney* (Berkeley, Calif.).

Jacobs, Robert B. On the propagation of a disturbance through a viscous liquid flowing in a distensible tube of appreciable mass. Bull. Math. Biophys. 15, 395–409 (1953).

The author criticizes previous work by other investigators [see, e.g., Karreman, same Bull. 14, 327–350 (1952); 15, 109 (1953); these Rev. 14, 781] on the grounds that their mathematical simplifications eliminate important effects and that their assumptions concerning the elastic properties of the tube material are unreal. An investigation is presented which, it is claimed, will probably apply to most actual systems and will show all the previously neglected effects. The analysis is based on a transformation of variables which renders the physically complicated boundary conditions simple, and on a subsequent expansion of the unknowns in power series in R_0/L (R_0, L the unstressed radius and length, respectively, of the tube) the first essential terms of which are found. Curves showing the variation of the propagation velocity of the fundamental mode of traveling waves with various parameters are given.

This reviewer questions the validity of the analysis for various reasons. (i) The elastic law assumed involves only one elastic constant, none analogous to Poisson's ratio for a linearly elastic material. Hence the possibly important effects of axial displacement of the tube wall as a result of circumferential strain are lost. (ii) The frictional force exerted by the fluid on the tube wall is neglected [see the paper reviewed above]. (iii) It is assumed that the effect of the steady flow upon which waves are superposed is adequately accounted for by replacing the nonlinear terms $u\partial u/\partial x+v\partial u/\partial r$ in the axial equation of motion (x, r; u, v axial and radial coordinates and velocity components respectively) by the single linear term $q_0\partial q^{(1)}/\partial x$, q_0 being the average steady velocity (a constant) and $q^{(1)}$ the axial perturbation. It is not at all clear that this assumption is warranted. Firstly, the term arising from $v\partial u/\partial r$ as a result of the radial variation of the steady flow U, say, may in certain circumstances be as important as the term retained. Secondly, the effect of the radial variation of the steady velocity U may be important also in the term $U\partial q^{(1)}/\partial x$. (iv) The parameter $\delta=R_0/L$ in powers of which solutions are sought, subject to the restriction $\delta\ll 1$, does not have any bearing on the study of traveling waves for which the tube must be assumed to have infinite length. The relevant ratio is R_0/λ, where λ represents the wave length of the disturbance [see Morgan and Kiely, loc. cit.]. (v) The reviewer cannot understand why the first terms of the power series in δ for the unknowns (zero power of δ) represent the steady flow. (vi) The derivation of the equations for the first important coefficients of the power series appears to be inconsistent with the assumptions. *G. W. Morgan.*

Imai, Isao. A new method of solving Oseen's equations and its application to the flow past an inclined elliptic cylinder. Proc. Roy. Soc. London. Ser. A. 224, 141–160 (1954).

In this paper is developed a general method of solving Oseen's linearized equations for a two-dimensional steady flow of a viscous fluid past an arbitrary cylindrical body.

The method is based on the fact that the velocity in the neighborhood of the cylinder can be generally expressed in terms of a pair of analytic functions, the determination of which from the appropriate boundary condition can be effected by successive approximations in powers of the Reynolds number, Re. The method enables one to obtain the velocity distribution near the cylinder and the lift and drag acting on it in the form of power series in Re, without recourse to manipulation of higher transcendental functions such as Bessel and Mathieu functions for circular and elliptic cylinders, respectively.

As an example of the application of the method, the uniform flow past an elliptic cylinder at an arbitrary angle of incidence is considered. Analytical expressions for the lift and drag coefficients are obtained which are correct to the order of Re, the lowest order terms being $O(\mathrm{Re}^{-1})$; and numerical calculations are carried out for the thickness ratio $t=0, 0.1, 0.5, 1$ and the Reynolds number $\mathrm{Re}=0.1, 1$. It is found that drag increases slightly with increase of either thickness-ratio or angle of incidence, and that lift decreases with increase of thickness-ratio while, as a function of the angle of incidence, it has a maximum at about 45°. (Author's abstract.) *Y. H. Kuo* (Ithaca, N. Y.).

Kuo, Y. H. On the flow of an incompressible viscous fluid past a flat plate at moderate Reynolds numbers. J. Math. Physics **32**, 83–101 (1953).

For large Reynolds number (Re) the problem of steady two-dimensional viscous flow can be treated by means of the Prandtl boundary layer. To deal with moderate Reynolds numbers it therefore seems appropriate to adopt a perturbation scheme based on the ordinary boundary-layer solution as first approximation. This the author has done for a flat plate of finite length in the flow direction.

Part I of the paper is devoted to investigation of second order effects in the boundary layer (which are $O(\mathrm{Re}^{-1})$ for the horizontal velocity component and $O(\mathrm{Re}^{-1})$ for the vertical component). The procedure involves determination of the modified potential flow outside the boundary layer due to the Blasius solution, from which the appropriate boundary conditions for the second approximation are then derived. The complete solution of the second-order viscous flow leads to a law of resistance which coincides with the Blasius law for $\mathrm{Re}>10^4$; in the range $15<\mathrm{Re}<10^4$ it agrees with experimental measurements of Janour.

In Part II it is shown that all approximations higher than the second will be singular at the leading edge in such a way that their contribution to resistance is undefined [cf. Alden, same J. **27**, 91–104 (1948), where a similar result is found for the semi-infinite plate; these Rev. **10**, 76]. Following an idea of Lighthill, this difficulty is overcome by perturbing the arguments of the solution, so that in the transformed system no objectionable singularities arise. The improved solution is singular only at the leading edge, whereas the Blasius solution is singular on a whole line, and it is found to satisfy the Stokes equation of slow motion in the vicinity of the leading edge. This improved representation of the leading edge is shown not to affect the plate resistance as calculated in Part I. *J. B. Serrin* (Minneapolis, Minn.).

Stewartson, K. Further solutions of the Falkner-Skan equation. Proc. Cambridge Philos. Soc. **50**, 454–465 (1954).

In this paper the Falkner-Skan equation

$$f'''+ff''+\beta^2(1-f'^2)=0$$

(f is a function of a single variable η and β is a parameter) [Philos. Mag. (7) **12**, 865–896 (1931)] is further examined. It is shown that (i) if $\beta<-0.1988$, all solutions subject to the boundary conditions in the case of a solid wall would have a slope $f'>1$ for some range of η and that (ii) f' approaches -1 as η tends to infinity. (iii) If $0>\beta>-0.1988$, there are two families of solutions. One agrees with Hartree's solutions [Proc. Cambridge Philos. Soc. **33**, 223–239 (1937)] and the new one has $f'<0$ in a small range of η. This family of solutions corresponds to a boundary-layer flow with separation. It is shown also that if $-0.5<\beta<0$, there is a family of solutions satisfying boundary conditions in the case of a free streamline. *Y. H. Kuo* (Ithaca, N. Y.).

Tani, Itiro. On the approximate solution of the laminar boundary-layer equations. J. Aeronaut. Sci. **21**, 487–495, 504 (1954).

Görtler, H. On the three-dimensional instability of laminar boundary layers on concave walls. NACA Tech. Memo. no. **1375**, 32 pp. (1954).

Translated from Nachr. Ges. Wiss. Göttingen. Fachgruppe I. (N.F.) **2**, 1–26 (1940); these Rev. **2**, 267.

Birkhoff, Garrett, et Kampé de Fériet, Joseph. Sur un modèle de turbulence homogène isotrope. C. R. Acad. Sci. Paris **239**, 16–18 (1954).

Les auteurs considérent un champ de vitesses aléatoire, associé à une turbulence homogène, isotrope et incompressible. Sous certaines hypothèses générales, ils démontrent que, étant donné un spectre d'énergie absolument continu $\phi(\mathbf{x})$, il lui correspond, dans l'espace de mesure Ω, une mesure μ et une seule définissant une statistique selon la loi normale. Cette mesure possède la transitivité métrique, d'où l'équivalence entre les moyennes spatiales et les moyennes statistiques. Enfin, si la vitesse évolue dans le temps suivant l'équation de la diffusion, et si, à l'instant $t=0$, une mesure μ du type considéré correspond au spectre $\phi(\mathbf{x})$, on peut, à un instant ultérieur $t>0$, trouver une mesure de même nature, et cette mesure est associeé au spectre $\exp(-2\nu^2 t)\phi(\mathbf{x})$. *J. Bass* (Chaville).

Davies, R. W. Energy spectrum of turbulence for the entire range. Physical Rev. (2) **95**, 912–913 (1954).

Dans un milieu turbulent homogène, isotrope, incompressible, le transfert d'énergie se fait par superposition de deux processus, l'un $d\hat{u}$ à l'inertie, l'autre $d\hat{u}$ à la viscosité. On connaît les fonctions spectrales limites $F_1(k)$, $F_2(k)$ de la turbulence, correspondant à chacun de ces deux processus, lorsque l'autre est négligeable:

$$F_1(k)\sim(\langle\epsilon\rangle)^{2/3}k^{-5/3},\quad F_2(k)\sim Ck^4\exp\{-14\nu k^2\langle u^2\rangle/\langle\epsilon\rangle\},$$

où $\langle u^2\rangle$ est l'énergie moyenne par unité de volume, et $\langle\epsilon\rangle$ l'énergie moyenne dissipée par unité de volume et de temps.

L'auteur se propose de construire une fonction spectrale $F(k)$ valable dans le cas général. Il suppose que, pour une variation donnée de $\langle\epsilon\rangle$, la variation de $F(k)$ provoquée par l'un des processus ci-dessus est en un certain sens indépendante de la variation provoquée par l'autre processus. Ces hypothèses le conduit, par des calculs dont la validité ne semble pas absolument concluante au rapporteur, à une forme plausible de $F(k)$. *J. Bass* (Chaville).

Ertel, Hans. Der Symmetriecharakter des Turbulenz-(Austausch-) Tensors. S.-B. Deutsch. Akad. Wiss. Berlin. Kl. Math. Allg. Nat. **1953**, no. 2, 8 pp. (1953).

Soit ϕ un certain champ scalaire associé à un écoulement turbulent. Le mécanisme de transfert de la quantité ϕ dans le mouvement turbulent est caractérisé par un "vecteur courant" S_i.

Le vecteur S_i se déduit de grad ϕ par une transformation linéaire $S_i=-\mu_{ik}\partial\phi/\partial x_k$, où μ_{ik} est un tenseur qui, dans le cas de la turbulence isotrope, prend la forme réduite $\mu\delta_{ik}$.

La question se pose de savoir si un tenseur symétrique ne se réduisant pas à $\mu\delta_{ik}$ est une représentation suffisante de la turbulence non isotrope, où si, plus généralement, il est nécessaire de faire appel à un tenseur non symétrique. L'auteur expose pourquoi, selon lui, μ_{ik} doit être symétrique.

J. Bass (Chaville).

Frenkiel, F. N. Effects of wire length in turbulence investigations with a hot-wire anemometer. Aeronaut. Quart. **5**, 1–24 (1954).

L'auteur étudie l'influence, sur les mesures classiques en soufflerie, de la longueur d'un fil chaud par ailleurs parfait et compensé. Il utilise des moyens temporelles prises sur de grands intervalles de temps. Il suppose que la turbulence est homogène autour du fil, et que son intensité est faible. R_x et R_y désignent les fonctions de corrélation longitudinale et transversale entre les composantes longitudinales de la vitesse en deux points. $L_x=\int_0^\infty R_x(s)ds$ et $L_y=\int_0^\infty R_y(s)ds$ sont les deux longueurs de corrélation.

Le fil chaud est placé perpendiculairement au courant moyen. Si l'on néglige la longueur l du fil, la différence de potentiel e aux bornes du fil est de la forme $e=Ku'$, u' étant la composante longitudinale de la vitesse turbulente, et K restant constant au cours d'une expérience. Si la longueur du fil est prise en considération, e est remplacé par $e_l=k\int_0^l u'(s,t)ds$, $K=kl$. L'auteur applique cette formule à la mesure de diverses moyennes classiques: (1) Energie longitudinale moyenne: On trouve

$$\overline{e_l^2}=\frac{\overline{e^2}}{I_l^2},\quad \text{avec}\quad \frac{1}{I_l^2}=\frac{2}{l}\int_0^l(l-s)R_y(s)ds.$$

Cas de la turbulence isotrope, en supposant l/L_y grand, petit, ou de l'ordre de l'unité (avec des formes analytiques simples pour R_y). (2) Fonctions de corrélation en deux points de la vitesse longitudinale, mesurées avec deux fils chauds parallèles normaux à la vitesse moyenne. Cas de la turbulence isotrope ou axisymétrique. Cas où l est petit par rapport à la distance des deux fils. (3) Longueurs L_x et L_y. (4) Longueur $\lambda=-1/R_x''(0)$.

J. Bass (Chaville).

Coburn, N. Discontinuities in compressible fluid flow. Math. Mag. **27**, 245–264 (1954).

Dans ce mémoire, l'auteur expose de façon complète et systématique la théorie des discontinuités en mécanique des fluides compressibles. Les équations générales de la théorie des caractéristiques, celles relatives aux ondes de choc et aux discontinuités de contact sont écrites dans les cas les plus généraux. Certes ces résultats en eux même sont en général bien connus. L'originalité de ce travail consiste sans doute en une formulation des lois de la mécanique des fluides parfaits comme lois de conservation dans un espace quadridimensionnel (3 variables cartésiennes et le temps); de façon plus précise certaines intégrales prises sur une hypersurface fermée arbitraire à 3 dimensions doivent être identiquement nulles. C'est ce point de départ qui permet l'élégant exposé de la théorie.

P. Germain (Paris).

Mollø-Christensen, Erik. Characteristic solution for axially symmetric transonic flow. J. Aeronaut. Sci. **21**, 501 (1954).

Legras, J. Nouvelles applications de la méthode de Lighthill à l'étude des ondes de choc. O.N.E.R.A. Publ. no. **66**, i+62 pp. (1953).

This report presents in complete detail the application of the "uniformly-valid" technique of Lighthill-Whitham in order to consider and compute approximations to the steady two-dimensional supersonic flow with shock waves about symmetric profiles at zero incidence, work the author has briefly reported upon previously [C. R. Acad. Sci. Paris **233**, 1005–1008 (1951); **234**, 1432–1434 (1952); these Rev. **13**, 701; **14**, 218]. The report also contains details of the author's previously reported work on the flow in the neighborhood of the junction of the plane shock and nearly conical shock at the wing tip of a rectangular wing in a supersonic stream [ibid. **234**, 181–183 (1952); these Rev. **13**, 701].

P. Chiarulli (Providence, R. I.).

Shibaoka, Yoshio. On the shock wave formed by a concave bend. J. Inst. Polytech. Osaka City Univ. Ser. B. Physics **4**, no. 5, 1–6 (1953).

The author has investigated the nature of the cusping of the envelope of the characteristics of a two-dimensional simple-wave flow along a smooth concave corner. The coordinate origin is taken as the point on the wall from which emanates the characteristic that passes through a cusp point and it is assumed that the equation of the wall is given by $y=x^2+ax^3$. The quantity a is then found as a simple function of μ_0, the Mach angle associated with the characteristic through the origin. A study of the second coefficients of a Taylor-series representation of the envelope in the neighborhood of the cusp point shows that, depending on the value of μ_0, the envelope branches out from the cusp point either toward increasing x and y values or toward decreasing x and y values. Computations are carried out for air ($\gamma=7/5$). It should be noted that the results with respect to the direction of branching are restricted to the case where the wall shape is exactly $y=x^2+ax^3$, i.e., this equation cannot be considered as an approximation to a power series representation of the wall shape in the neighborhood of the point of interest.

P. Chiarulli (Providence, R. I.).

Miles, John W. On the transformation of the linearized equation of unsteady supersonic flow. Quart. Appl. Math. **12**, 1–12 (1954).

The linearized potential equation for unsteady motion in frictionless, supersonic flow is transformed from the classical wave equation to the canonical form $\phi_{ss}-\phi_{yy}-\phi_{zz}=\phi_{rr}$ with the aid of a modified Lorentz transformation. Possible invariant transformations of the latter, including the classical Lorentz transformation, are discussed. Eleven coordinate systems (each of which has its counterpart in the classical theory of the wave equation) permitting separation of variables are set forth, their derivation being based on the analogy between the hyperbolic metric defined by $(ds)^2=(dx)^2-(dy)^2-(dz)^2$ and the Euclidean (Cartesian) metric. A few practical applications are indicated. (Author's summary.)

D. C. Pack (Glasgow).

Brown, W. F., and Thomas, T. Y. Limiting behavior of pressure derivatives behind shocks in supersonic gas flow. J. Rational Mech. Anal. **3**, 231–245 (1954).

Formulae for the values of various quantities immediately behind a shock-wave attached to the nose of a curved profile

are obtained in terms of ascending powers of the proportionate density increase across the shock-wave. When the independent variables are taken as S, the specific entropy, and ω, the inclination of the flow at any point to the direction of the incident flow, these formulae yield the limiting values at infinity (where the shock-wave becomes a Mach line) of the pressure derivatives $\partial p/\partial S$ and $\partial p/\partial \omega$ at the shock-wave. The authors suggest that the pressure at a point P of the profile may be computed approximately by an application of the first mean-value theorem along the curve $\omega=$constant between P and the corresponding point Q of the shock-wave, the value of $\partial p/\partial S$ appropriate for the theorem being replaced by the value of this derivative at Q. *D. C. Pack.*

Brown, W. F., and Thomas, T. Y. Rear shock and pressure on supersonic airfoils. Proc. Nat. Acad. Sci. U. S. A. **40**, 83–87 (1954).

In the paper reviewed above the authors developed a method for obtaining an approximation to the pressure on the front portion of a curved profile. The method is here extended to obtain the pressure distribution over the rear portion of the profile. The authors point out that use of the method is hampered by the difficulty of prescribing conditions behind the rear shock-wave and that the assumption made in the paper (a uniform stream behind this shock-wave with the Mach number of the incident stream), since it violates the second law of thermodynamics, is at best only a rough approximation. *D. C. Pack* (Glasgow).

Thomas, T. Y. Sonic point on supersonic airfoils. Proc. Nat. Acad. Sci. U. S. A. **40**, 76–83 (1954).

It is assumed that there is a shock-wave detached from the airfoil and that there is a streamline perpendicular to the shock-wave which is also the dividing streamline for the airfoil. If ω denotes the angle of inclination of the flow at any point to the incident stream and S the entropy, and if P, Q are the sonic points (in, say, the upper half-plane) on the airfoil and behind the shock-wave respectively, the author suggests that

$$\omega_P-\omega_Q=(d\omega/dS)_Q(S_P-S_Q)$$

is generally a good approximation. The quantities on the right-hand side are calculated, ω_Q is known, and hence the sonic point P of the airfoil is found. Some computations are included. *D. C. Pack* (Glasgow).

Lord, W. T., and Eminton, E. Slender bodies of minimum wave drag. J. Aeronaut. Sci. **21**, 569–570 (1954).

Weber, Sophus. Über den Zusammenhang zwischen der laminaren Strömung der reinen Gase durch Rohre und dem Selbstdiffusionskoeffizienten. Danske Vid. Selsk. Mat.-Fys. Medd. **28**, no. 2, 138 pp. (1954).

The title of this paper is perhaps chosen to emphasize the author's expressed view that the self-diffusion coefficient has a physical meaning in addition to just being a mathematical convenience and that it plays an important role in the theory of laminar flow through long narrow tubes. The title does not adequately represent the contents of this lengthy paper, however, for it actually includes an extensive review, interpretation and refinement of the semi-phenomenological theory of laminar flow through narrow tubes covering the whole range from free molecular flow to Poiseuille flow. The total flow is considered to be the sum of three parts, one due to self-diffusion (giving the free molecular flow at low densities), the second due to slip and the third to pure Poiseuille flow. Each of these contributions is considered separately in some detail. The monograph also includes a 15 page summary. *G. Newell.*

Herbeck, M. Approximate solutions for heat transfer with convection flows. J. Aeronaut. Sci. **21**, 142–144 (1954).

Solutions of the heat transfer equation

$$(1)\qquad (\mu/\sigma)T_{yy}=\psi_yT_x-\psi_xT_y,$$

where μ, σ are constant, and ψ, the stream function, is of the form (2) $\psi=y^{r-1}f(x)$, especially when

$$(3)\qquad \psi=\text{const.}\times y^{r-1}x^{r+1},$$

are obtained as expressions involving confluent hypergeometric functions. For the temperature field of a heated wall considered by S. Levy [same J. **19**, 341–348 (1952)], who obtained a solution by numerical integration with the exact form of ψ, the author shows that the use of the approximate stream function, which is of the kind (3) above, gives results in error by about 5%. *D. S. Gilles.*

Kaplan, S. A. A system of spectral equations of magneto-gas-dynamic isotropic turbulence. Doklady Akad. Nauk SSSR (N.S.) **94**, 33–36 (1954). (Russian)

In this paper an attempt is made to generalize Heisenberg's theory of turbulence [Z. Physik **124**, 628–657 (1948); these Rev. **11**, 63] to the case when the fluid considered is an infinitely good electrical conductor and spontaneously generated turbulent magnetic fields are possible. Just as Heisenberg replaced the transfer term representing the exchange of energy between the different Fourier components by an expression involving the spectrum of turbulence, the author replaces the terms representing the interaction of the velocity field with the magnetic field (the term $\mathbf{j}\times\mathbf{H}$ in the equation of motion and the term curl $(\mathbf{u}\times\mathbf{H})$ in the equation for H) by similar heuristic expressions involving the spectrum of $\mathbf{u}$ and of $\mathbf{H}$. The paper is qualitative in its contents. *S. Chandrasekhar* (Williams Bay, Wis.).

Kaplan, S. A., and Stanyukovič, K. P. Solution of the equations of magneto-gas-dynamics for one-dimensional motion. Doklady Akad. Nauk SSSR (N.S.) **95**, 769–771 (1954). (Russian)

It is known that in one dimension, disturbances of finite amplitude in a gas in which the pressure p and the velocity u are functions of the density ρ only, are propagated in such a way that u and ρ are constant at points which move with the speed $u+c$ where c $(=(dp/d\rho)^{\frac{1}{2}})$ is the local velocity of sound. Considering the standard equations of hydromagnetics in one dimension in which the magnetic field $\mathbf{H}$ is also a function of ρ, the author shows that the classical result continues to hold if the local velocity of sound is replaced by $(c^2+V_A^2)^{\frac{1}{2}}$ where $V_A=H(\mu/4\pi\rho)^{\frac{1}{2}}$ is the Alfvén velocity. *S. Chandrasekhar* (Williams Bay, Wis.).

Crausse, Etienne. Similitude hydrodynamique d'écoulements liquides soumis à des actions électromagnétiques. C. R. Acad. Sci. Paris **239**, 151–152 (1954).

Elsasser, Walter M. Dimensional relations in magneto-hydrodynamics. Physical Rev. (2) **95**, 1–5 (1954).

In this paper an attempt is made to give precision to a number of general ideas currently prevalent in the applications of hydrodynamics to bodies of cosmical dimensions in which there are prevalent magnetic fields. Thus the fact that in the equation governing the magnetic field we may

neglect the term representing the displacement current can be traced to the large value of the magnetic Reynolds number $R_m=\mu\sigma\lambda V$ where μ and σ are the coefficients of magnetic permeability and electrical conductivity (in m.k.s. units) and λ and V are, respectively, measures of the linear dimension of the system and of the velocities present. Similarly, the fact that the conduction current (resulting from a space-charge density) can be ignored is due to the smallness of $(\epsilon V/\lambda\sigma)$ where ϵ denotes the dielectric constant. And the space charge density can itself be ignored when the magnetic Reynolds number is large. The often quoted statement that in an infinitely good electrical conductor the lines of force may be considered as attached to the fluid is only another way of stating that the rate of change of the magnetic induction across any surface (moving with the fluid) is given by $1/\sigma$ times the line integral of the current along the bounding curve (analogue of the Helmholtz-Kelvin vorticity theorem). As stated, all these are implicit in the equations of motion and Maxwell's equations in the form in which they are currently used. And with the equations for $\mathbf{u}$ and $\mathbf{H}$, as approximated, the author shows that they satisfy symmetrical equations; and from this symmetry the equipartition (or at any rate an equality at least as to order of magnitude) of the kinetic energy of motion and the magnetic energy in the field is "deduced". Finally, the author considers the plausibility of theories (such as Fermi's) on the acceleration of cosmic ray particles in the galactic magnetic fields and concludes that the equality (as regards order of magnitude) of the energy density of the cosmic rays and the energy of peculiar motion of the interstellar matter suggests that cosmic radiation is in dynamical equilibrium with the galactic magnetic field. [This last conclusion was first reached by Fermi, Physical Rev. (2) 75, 1169–1174 (1949).] *S. Chandrasekhar.*

Mintzer, David. Wave propagation in a randomly inhomogeneous medium. III. J. Acoust. Soc. Amer. 26, 186–190 (1954).

Dans la première partie de son travail [même J. 25, 922–927 (1953); ces Rev. 15, 481] l'auteur a introduit l'hypothèse restrictive que l'indice de refraction en un point quelconque du milieu ne change pas pendant le temps de passage de l'impulsion à travers ce point, mais change seulement entre deux impulsions. Dans le présent travail l'auteur essaie de lever cette restriction en introduisant une corrélation entre une impulsion et celle qui la suit. L'auteur suppose que la fonction de corrélation de l'hétérogénéité de l'indice de refraction peut être représentée comme un produit de deux fonctions, l'une dépendant des coordonnées, l'autre du temps. L'auteur arrive de cette façon à trouver la valeur du coefficient de variation V^2 et de ϕ fonction de corrélation des signaux successifs. *M. Kiveliovitch.*

Obuhov, A. M. On the influence of weak inhomogeneities of the atmosphere upon the propagation of sound and light. Izvestiya Akad. Nauk SSSR. Ser. Geofiz. 1953, 155–165 (1953). (Russian)

L'auteur traite l'équation des ondes (1) $c^{-2}\partial^2\varphi/\partial t^2-\Delta\varphi=0$ où la vitesse de propagation: $c=c_0/n$, n l'indice de refraction voisin de l'unité $n=1+\mu$, $\mu\ll 1$ fonction des coordonnées du point. Considérons une symétrie sphérique et soit φ une onde monochromatique de fréquence ω

$$\varphi=A(r)\exp\{i[\omega t-S(r)]\}.$$

En substituant cette valeur de φ dans l'équation (1) et en négligeant les termes en μ^2, on trouve, en introduisant quelques hypothèses restrictives, les valeurs de $\log A(r)$ et de la phase $S(r)$. L'auteur reprend ensuite le problème du point de vue statistique en introduisant un coefficient de corrélation de la forme $B(M, M_1)=\mu(M)\mu(M_1)$, où M et M_1 sont deux points voisins, ce qui permet d'expliciter davantage les expressions $\log A(r)$ et $S(r)$. On applique les résultats obtenus aux problèmes: (a) la scintillation des étoiles et (b) les pulsations de l'amplitude du son qui se propage dans une atmosphère légèrement hétérogène.

M. Kiveliovitch (Paris).

Sorokin, V. S. On stability of a nonuniformly heated gas in a gravitational field. Akad. Nauk SSSR. Prikl. Mat. Meh. 17, 149–156 (1953). (Russian)

Il est bien connu qu'une couche de fluide non uniformément chauffée se trouve en équilibre mécanique lorsque le gradient de température est parallèle à la force de gravitation et que la stationnarité du courant thermique est assurée. Mais l'équilibre n'est pas toujours stable et on se trouve en présence d'une convection lorsque le gradient de température dépasse une certaine valeur.

La détermination de cette grandeur présente de grandes difficultés; pour faciliter le problème l'auteur étudie deux cas particuliers. Dans le premier cas on néglige la variation de densité provenant des différences des pressions aux différents niveaux par rapport aux variations de densité provenant des différences des températures. Dans ce cas les recherches de Rayleigh et d'autres ont permis de déduire une certaine inégalité qui permet de préciser cette valeur critique. Un autre cas intéressant est celui où l'on néglige dans les équations les influences de la viscosité et de la conductibilité thermique. Ce cas n'a pas encore été étudié d'une manière rigoureuse. L'auteur reprend la question et montre que le critère obtenu confirme la formule donnée il y a déjà bien longtemps par Schwarzschild. *M. Kiveliovitch.*

Rogers, M. H. The forced flow of a thin layer of viscous fluid on a rotating sphere. Proc. Roy. Soc. London. Ser. A. 224, 192–208 (1954).

The paper is concerned with the motion, due to differential heating, of a thin layer of viscous fluid on a rotating sphere. The motion is assumed to be steady and zonally uniform, and the results are compared with the general circulation of the atmosphere when averaged over a long period of time. A constant coefficient of eddy-viscosity is used, and heat is assumed to be transferred solely by the process of eddy-conduction.

The equations governing the motion are first reduced by making use of the fact that the depth of the fluid is small compared with the radius of the sphere. The resulting set of non-linear equations is then solved by the method of expansion in ascending powers of a small parameter, giving a linear set of equations at each stage. The small parameter in question is the Rossby number for the flow pattern, which is the ratio of the maximum relative zonal velocity of the fluid to the equatorial velocity of the earth. The method thus displays the effect of the inclusion of the non-linear terms on large-scale atmospheric motions. The results indicate that these terms do not significantly alter the flow pattern in middle and high latitudes; but in low latitudes they completely change the character of the motion, giving a zonal flow from east to west in latitudes 0–15° N., a region of ascending air centered at 25° N. and a surface belt of high pressure with a maximum at 10° N. Finally, it is shown that, if friction is omitted, the belt of easterlies disappears,

indicating that a momentum transfer process is necessary in order to maintain the trade-wind belt.

G. C. McVittie (Urbana, Ill.).

Elasticity, Plasticity

Sneddon, Ian N. Quelques solutions des équations du mouvement d'un solide élastique. Rend. Circ. Mat. Palermo (2) **3**, 115–129 (1954).

This paper records a partly expository address delivered by the author. The opening portion is devoted to a general discussion of the use of integral transforms in the solution of the equations of motion appropriate to a linear, homogeneous, isotropic, elastic medium. There follows a formal treatment of the special case presented by a medium occupying the entire space under the action of a parallel time-dependent body-force field by means of Fourier transforms and, in the axisymmetric case, by means of Hankel transforms; the solution corresponding to a single concentrated force, whose magnitude is a function of time, is reached with the aid of the Dirac delta-function. The paper closes with a list of special problems which the author has solved on the basis of integral-transforms, and a detailed treatment of which is to appear in a future publication. The problems cited here refer to the entire space, the half-space, and the body bounded by two parallel planes under various particular time-dependent systems of body forces and surface tractions. *E. Sternberg* (Chicago, Ill.).

Gol'denveĭzer, A. L. On the calculation of shells with concentrated forces. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 181–186 (1954). (Russian)

There are two methods of calculating shells on concentrated forces. The first one starts with a distributed load acting in a small region which is allowed to shrink to a point, the load accordingly increasing infinitely at the same time. The second method consists of constructing a function satisfying the elasticity differential equations which has a certain defined singularity in the neighborhood of the point of application of the concentrated force. The author considers the second method only, which is mathematically very convenient, but which can be used only if the nature of the singularity is known beforehand. The author uses the following singularity: $r^2 \ln r$. *T. Leser.*

Ambarcumyan, S. A. On the construction of approximate theories of the computation of sloping cylindrical shells. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 303–312 (1954). (Russian)

This paper gives a systematic collection of all known approximate methods for solving the elasticity problem of sloping cylindrical shells. These methods are scattered throughout the literature which is listed at the end of the paper. The choice of a given method depends chiefly on the ratio of dimensions. The approximate methods are based on the following assumptions and simplifications: (a) that longitudinal bending moments are absent; (b) that deformations along a cylindrical element of the middle surface are absent; (c) that turning moments are absent; (d) that in the middle surface the torsional displacements are absent; (e) that bending moments in the plane of the circular cross-section are absent; (f) that deformations along an arc of the middle surface are absent.

For a certain ratio of dimensions a certain combination of two or three of these assumptions is used and the differential equations simplify accordingly. The author discusses all possible cases, solves an example for a given external load and given supports, and also presents tables with numerical data for different ratios of dimensions.

T. Leser (Lexington, Ky.).

Solomon, L. A one-dimensional problem for a helicoidal shell. Akad. Nauk SSSR. Prikl. Mat. Meh. **18**, 43–54 (1954). (Russian)

This work is based on results and methods of U. N. Rabotnov [same journal **10**, 639–646 (1946); these Rev. **8**, 359]. Rabotnov showed the advantage of using equations in asymptotic form for surfaces with negative Gaussian curvatures. The middle surface of the shell considered by the author is a minimal helicoid (the asymptotic curves form an orthogonal set). The helicoid extends without limit in all directions and the conditions of support and the loads depend only on x which is the distance from z-axis. Under such conditions the strains, stresses and displacements will depend also on x only. The differential equations reduce to ordinary differential equations. The author solves the above elasticity problem and compares his results with the results of several others. *T. Leser* (Lexington, Ky.).

Aggarwala, B. D. Singularly loaded rectilinear plates. I. Z. Angew. Math. Mech. **34**, 226–237 (1954). (German, French and Russian summaries)

This paper discusses the general problem of the small transverse deflections of simply-supported, polygonal, elastic plates under the action of concentrated loads. The analysis required bears a close resemblance to that involved in the determination of the stream function of incompressible non-viscous two-dimensional flow within a region due to a line vortex, the vortex being situated at the point of application of the concentrated load and the boundary of the region being identical with that of the plate. An approach via the theory of functions of a complex variable is therefore available, and it is shown that several interesting particular cases may be elegantly discussed.

H. G. Hopkins (Sevenoaks).

Contri, Lorenzo. Studio sulla lastra di spessore variabile linearmente nella direzione di una coppia di lati. Ist. Veneto Sci. Lett. Arti. Atti Cl. Sci. Mat. Nat. **111**, 183–195 (1953).

This paper concerns the small transverse displacements of a rectangular plate of variable thickness. Two opposite edges are restricted to be simply supported but the boundary conditions along the other two edges may be quite general. The thickness varies linearly across a section perpendicular to the simply supported edges and is constant across a section parallel to these edges. The fundamental differential equation admits of a relatively simple treatment when Poisson's ratio ν has the value 1/3, and this special case is discussed generally for various types of applied load including that of uniform load. An iterative method of successive approximation is suggested to deal with the case $\nu \neq 1/3$. *H. G. Hopkins* (Sevenoaks).

Müller, W. Zur Biegungstheorie einer Vierpilzplatte mit rechteckigen Stützflächen. Ing.-Arch. **22**, 163–170 (1954).

This paper gives an extension of previous work by the author [see Ing.-Arch. **22**, 60–72 (1954); these Rev. **15**, 842]

on the analysis of mushroom floors. The loads applied to the floor are now uniformly spread out over rectangular-shaped areas instead of being concentrated at isolated points. *H. G. Hopkins* (Sevenoaks).

***Yu, Yi-Yuan. Solution for the exterior of a general ovaloid under arbitrary loading and its application to square rigid core problems.** Proceedings of the First U. S. National Congress of Applied Mechanics, Chicago, 1951, pp. 227–237. The American Society of Mechanical Engineers, New York, N. Y., 1952.

Muskhelishvili's complex-variable analysis is applied to solve certain plane-strain and generalised plane-stress problems of elasticity theory. These problems all involve a rigid core which fills up a hole, shaped like a square with rounded corners, in elastic material, a perfect bond between the rigid and elastic material being assumed. Solutions are given when the elastic material is (a) infinite in extent, the stresses at infinity being constant or the core being subjected to an isolated force and couple; and (b) shaped like a beam with the core center situated symmetrically, the beam being subject to pure bending stresses at infinity. Illustrative numerical results for the stresses at the core boundary are given for all cases considered. *H. G. Hopkins.*

Mossakovskiĭ, V. I., and Zagubiženko, P. A. On a mixed problem of the theory of elasticity for a plane weakened by a rectilinear gap. Doklady Akad. Nauk SSSR (N.S.) **94**, 409–412 (1954). (Russian)

The method of solution of plane problems in elasticity developed by Mushelišvili is used to solve a particular problem of deformation of an elastic plane weakened by a rectilinear crack when the plane is subjected to the action of forces of constant intensity making an arbitrary angle with the direction of the crack. *I. S. Sokolnikoff.*

Podstrigač, Ya. S. Stresses in a plane weakened by two unequal circular openings. Dopovidi Akad. Nauk Ukrain. RSR **1953**, 456–460 (1953). (Ukrainian. Russian summary)

Bipolar coordinates are used to investigate the stress concentration in an infinite elastic plane weakened by two unequal circular openings. Special consideration is given to two examples: (a) the circular contours are subjected to uniform but unequal pressures; (b) the plane is in the state of bi-axial tension and the circular contours are free of external forces. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Mecugov, V. H. On bending by a couple of a stretched prismatic beam composed of various elastic materials. Soobščeniya Akad. Nauk Gruzin. SSR **14**, 459–465 (1953). (Russian)

Ruhadze, A. K. The problem of bending by a couple of a bar composed of various elastic materials and with a slightly bent axis. Soobščeniya Akad. Nauk Gruzin. SSR **14**, 525–532 (1953). (Russian)

Gorgidze, A. Ya. On the problem of bending by a transverse force of a stretched prismatic bar composed of various materials. Soobščeniya Akad. Nauk Gruzin. SSR **14**, 589–594 (1953). (Russian)

Gorgidze [same Soobščeniya **13**, 73–80 (1952); these Rev. **15**, 75], Ruhadze [ibid. **13**, 137–144, 265–272 (1952); these Rev. **15**, 75], and Šarangiya [ibid. **13**, 389–396 (1952); these Rev. **15**, 75] have considered several problems of deformation of the initially twisted composite beams. The same methods are used in these papers to solve the problems stated in the titles. *I. S. Sokolnikoff.*

Hatiašvili, G. M. On deformation of a composite cylindrical bar with a lateral load varying along the generators of the cylinder. Soobščeniya Akad. Nauk Gruzin. SSR **14**, 197–204 (1953). (Russian)

This is an extension of an earlier paper [same Soobščeniya **13**, 335–341 (1952); these Rev. **15**, 75] covering the case of external loads of the form $T_i=\sum_{k=0}^{n}A_{ki}(x_1, x_2)x_3^k$, $i=1, 2, 3$, where x_3 is measured along the axis of the cylinder. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Razmadze, G. N. On torsion of circular shafts of variable cross-section due to impact. Soobščeniya Akad. Nauk Gruzin. SSR **14**, 91–94 (1953). (Russian)

The note contains a brief consideration of those solutions of the system of vibration equations

$$\mu\nabla^2 u_\alpha+(\lambda+\mu)\frac{\partial\vartheta}{\partial x_\alpha}=\rho\ddot{u}_\alpha, \quad \alpha=1, 2,$$

$$\frac{\partial\vartheta}{\partial x_3}=0, \quad \vartheta\equiv\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3},$$

that are of interest in the study of torsional vibrations of axially-symmetric rods. *I. S. Sokolnikoff.*

Kumar, S., and Joga Rao, C. V. A rotating disc in constant pure shear. J. Indian Inst. Sci. **36**, 102–106 (1954).

Payne, L. E. On axially symmetric crack and punch broplems for a medium with transverse isotropy. Proc. Cambridge Philos. Soc. **50**, 466–473 (1954).

In this paper the author, following the work of Green [same Proc. **45**, 251–257 (1949); these Rev. **10**, 649] and Shield [ibid. **47**, 401–409 (1951); these Rev. **12**, 878] presents a straightforward method for solving the general axially symmetric crack and punch problems for media with transverse isotropy. He first interprets the displacements and stresses in terms of a potential function whose value or derivative is given on the boundary, and then by means of an appropriate system of curvilinear coordinates the solutions to the problems with transverse isotropy follow immediately from the solutions to the corresponding isotropic problems, which have been given in a previous paper [J. Soc. Indust. Appl. Math. **1**, 53–71 (1953); these Rev. **15**, 267]. *R. M. Morris* (Cardiff).

Hu, Hai-chang. On the three-dimensional problems of the theory of elasticity of a transversely isotropic body. Acta Sci. Sinica **2**, 145–151 (1953).

The author obtains a general solution of the equilibrium equations of linear elasticity for materials possessing transverse isotropy which is complete. References to other papers dealing with this problem and a rather elegant general solution for axially symmetric problems may be found in a paper by Eubanks and Sternberg [J. Rational Mech. Anal. **3**, 89–101 (1954); these Rev. **15**, 482]. *J. L. Ericksen.*

Miller, G. F., and Pursey, H. The field and radiation impedance of mechanical radiators on the free surface of a semi-infinite isotropic solid. Proc. Roy. Soc. London. Ser. A. **223**, 521–541 (1954).

The authors follow Lamb [Philos. Trans. Roy. Soc. London. Ser. A. **203**, 1–42 (1903)] to obtain solutions to four boundary-value problems for a semi-infinite isotropic

elastic solid occupying the half-space $z>0$. The problems are: (a) a strip of finite width vibrating normally to the surface $z=0$; (b) a strip of finite width vibrating tangentially to $z=0$ and normally to the axis of the strip; (c) a circular disk vibrating normally to $z=0$; and (d) a disk performing tangential vibrations about its center. [See also the book by V. D. Kupradze, Boundary problems of the theory of vibrations and integral equations, Gostehizdat, Moscow-Leningrad, 1950; these Rev. **15**, 318.] Explicit asymptotic formulae for large distances from the vibrator for each of the four cases are obtained. *C. R. De Prima.*

Bishop, R. E. D., and Goodier, J. N. On Eulerian coordinates in elastic wave propagation. J. Mech. Phys. Solids **2**, 103–109 (1954).

The authors observe that some dynamical problems in linear elasticity, such as those involving a wave propagating with constant velocity and without change of form, can be solved quite easily by transforming to a coordinate system moving with the disturbance. Other authors, e.g., Sáenz [J. Rational Mech. Anal. **2**, 83–98 (1953); these Rev. **14**, 601], have realized this. *J. L. Ericksen.*

Duffin, R. J., and Schild, A. The effect of small constraints on natural vibrations. Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 155–163. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00.

Consider a linear mechanical system subject to external constraints which can be varied continuously (for example, a clamped plate whose boundary can be deformed). The authors present Theorem A: The infinitesimal increase in an eigenvalue is equal to the work done by a certain force (equal to the amplitude of the constraining force in the normalized vibration of the system with infinitesimally varied constraints) in statically deflecting the original system. A proof of this theorem is outlined, but a full demonstration is deferred to a later paper. The authors also introduce Postulate B: "Close to a rigid constraint, the stiffness of an elastic body is essentially independent of other constraints which are far away." (Cf. Saint-Venant's Principle.) The authors analyse the vibrating string and compare the exact results with the results obtained by applying (A) and (B), thus illustrating the accuracy of (A) and (B). Next the authors apply (A) and (B) in order to indicate a proof that there exist clamped plates which have nodal lines in their lowest modes. [Cf. the remark in Weinstock, Calculus of variations, McGraw-Hill, New York, 1952, p. 259, line 1 [these Rev. **14**, 661] that this is still (1952) an unsolved problem. Cf. also the references cited in the paper under review and also the review of Duffin, J. Math. Physics **31**, 294–299 (1953); these Rev. **14**, 601.] The paper is concluded by a short section on the membrane and acoustic vibrations. *H. D. Block* (Minneapolis, Minn.).

Troesch, Andreas, Anliker, Max, and Ziegler, Hans. Lateral vibrations of twisted rods. Quart. Appl. Math. **12**, 163–173 (1954).

Analysis is given of the problem of the free, small, flexural vibrations of a uniform, heavy, vertical, elastic rod built-in at its lower end. The rod is uniformly twisted and is of rectangular cross-section. The fundamental equations of motion are derived, and the form of their solution obtained. In proceeding, attention is confined to the case of a perfectly-narrow cross-section, and detailed numerical results are presented. For small or large values of the total twist the natural frequencies are computed from simple expansion formulas. In the intermediate range the natural frequencies were calculated by means of the sequence controlled computer at E. T. H., Zurich. *H. G. Hopkins.*

Dolph, C. L. On the Timoshenko theory of transverse beam vibrations. Quart. Appl. Math. **12**, 175–187 (1954).

This paper mainly concerns the normal modes of a uniform beam hinged at both ends. For each mode, there are two possible vibration frequencies. These are tabulated for values of two parameters which measure the importance with respect to the flexural rigidity of the sheer rigidity and of the rotary inertia. (The inclusion of the last two factors distinguishes Timoshenko's theory from the elementary one.) *D. R. Bland* (Newcastle-on-Tyne).

Miklowitz, Julius. Flexural waves in beams according to the more exact theory of bending. U. S. Naval Ordnance Test Station, Inyokern, Calif., Navord Rep. **2049**, vi+58 pp. (1953).

Part I of this report [which appears with minor alteration in J. Appl. Mech. **20**, 511–514 (1953)] presents a method of deriving wave solutions for the Timoshenko theory of the dynamics of beams including shear and rotatory inertia effects. The system of coupled equations for bending and shear deflections, which was given by Anderson [ibid. **20**, 504–510 (1953)], is transformed by the Laplace transform, and the transforms for deflections, moments and shears are obtained. The problem of the infinite beam with a concentrated transverse, time-dependent load at the origin, which had been treated previously by Uflyand [Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 287–300 (1948); these Rev. **10**, 88] and Dengler and Goland [Proc. First U. S. Nat. Congress Appl. Mech., 1951, Amer. Soc. Mech. Engrs., New York, 1952, pp. 179–186; these Rev., **13**, 820], is discussed. The author states that the following three significant results are shown: (1) that the moment transform obtained agrees with the Dengler and Goland results; (2) that the Uflyand solution, when interpreted properly, has meaning; and (3) that (wave-type) solutions to the semi-infinite beam and to certain finite beam problems can be produced.

In Part II, the method is used to derive the bending stress wave solution to the problem of a pure moment (taken as a half-sine wave pulse) applied to the free end of a semi-infinite beam. Inversion of the transforms is not elementary. Curves representing the results are presented, along with a discussion of the numerical integration problems involved in deriving them. Curves are compared with those from elementary beam theory and with others obtained by an electrical analog computer. *W. Nachbar.*

Micheletti, Valerio B. Vibrazioni di una trave semplicemente appoggiata soggetta all'azione concentrata di una massa, una molla ed un ammortizzatore. Atti Accad. Ligure **9** (1952), 283–313 (1953).

This paper concerns certain problems involving the free or forced flexural vibrations of simply supported elastic beams. A typical example is the case of a beam carrying a concentrated mass acted upon by a spring and dash-pot in parallel. Numerical results are given for the natural frequencies of certain systems. *H. G. Hopkins.*

Satô, Yasuo. Study on surface waves. X. Equivalency of SH-waves and sound waves in a liquid. Bull. Earthquake Res. Inst. Tokyo **32**, 7–16 (1954). (Japanese summary)

Melan, E. Wärmespannungen infolge eines rotierenden Temperaturfeldes. Österreich. Ing.-Arch. **8**, 165–170 (1954).

The determination of the thermal stress field in a thin circular plate is considered for the case of a steady temperature field. The analysis incorporates the concepts of the theory of generalised plane stress. The problem arises in the design of gas-turbine blades. *H. G. Hopkins.*

Hieke, Max. Über ein ebenes unstetiges Temperaturspannungsproblem. Z. Angew. Math. Mech. **34**, 121–139 (1954). (Russian summary)

Attention is drawn to problems concerning the determination of the stress and displacement fields induced in a homogeneous isotropic elastic medium by a steady temperature field which is discontinuous. As an example a complete solution is given of the following plane strain problem. Let r, θ be polar co-ordinates taken in a plane of plane strain. Then elastic material occupies an infinitely-long circular cylindrical region whose surface $r=r_0$ is stress free. There is unit temperature throughout the sectorial region $0\leqq r\leqq r_0$, $0\leqq|\varphi|<\varphi_0<\pi$, and zero temperature elsewhere. The problem is formulated as the determination of a stress function. Numerical values of the stresses at the temperature discontinuity surface are given for the case $\varphi_0=\frac{1}{3}\pi$. *H. G. Hopkins* (Sevenoaks).

Prager, W. Three-dimensional plastic flow under uniform stress. Rev. Fac. Sci. Univ. Istanbul (A) **19**, 23–27 (1954).

Particular solutions in the field of plane flow of a perfectly plastic body are known. Although the theory of three-dimensional flow of such a body has been discussed by T. Y. Thomas [J. Rational Mech. Anal. **1**, 343–357 (1952); **2**, 339–381 (1953); these Rev. **14**, 113, 820], B. Finzi [Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **76**, 222–238 (1941); these Rev. **7**, 501] and R. Hill [The mathematical theory of plasticity, Oxford, 1950; these Rev. **12**, 303], no particular solutions have been studied in detail. The present author constructs such solutions by considering a state of constant normal stress. If the rate of strain is proportional to the overstress (the von Mises relation), then, by eliminating the velocity components, the author finds that the proportionality factor satisfies either: (1) three partial differential equations (the regular case); or (2) four partial differential equations (the degenerate case). The velocity fields for both cases are determined. *N. Coburn.*

Onat, E. T., and Prager, W. The necking of a tension specimen in plane plastic flow. J. Appl. Phys. **25**, 491–493 (1954).

The v. Mises equations for plane plastic flow [R. Hill, The mathematical theory of plasticity, Oxford, 1950, Chapter VII; these Rev. **12**, 303] are simplified by a linearization of the yield condition about the stress state at yield in uniform tension. Using the linearized yield condition as the plastic potential, a linearized flow rule and a very simple, linear slip line field are obtained. The linearized relations are used to solve the problem of symmetrical necking in plane plastic flow for an almost prismatic tension specimen which is weakened by two long and extremely shallow, symmetrical grooves. The simplicity of the velocity field so obtained enables the finite plastic deformations during necking to be readily determined. *W. Nachbar.*

Hill, R. The plastic torsion of anisotropic bars. J. Mech. Phys. Solids **2**, 87–91 (1954).

The fully plastic state of stress is examined in the torsion of a prismatic bar composed of an anisotropic rigid-plastic material. If (x, y) axes are taken in some transverse direction, the z-axis being parallel to the generators, the state of stress in each element is to be a pure shear acting over the section and its (x, y)-components are denoted by (p, q). When the bar is fully plastic, these satisfy some relation $f(p^2, q^2, x, y)=0$, where f is the yield function and only the squares of p and q are involved because of the orthotropic symmetry assumed by the author. It is then shown how the distributions of shearing stress and warping displacement can be calculated without difficulty for any shape of section. Explicit formulas are obtained for the torque when the section is circular or rectangular and when the yield function is $f=p^2/p_0^2+q^2/q_0^2-1=0$, where p_0 and q_0 are the yield stresses in pure shear in the x and y directions respectively. The author points out that this equation probably represents fairly well the anisotropy in a specimen cut from the central portion of a rolled slab.

E. T. Onat (Ankara).

Craggs, J. W. Wave motion in plastic-elastic strings. J. Mech. Phys. Solids **2**, 286–295 (1954).

The author discusses the problem of obtaining approximate solutions of the equations of motion for a perfectly flexible extensible string, assuming that the tension is a function of the extension. The main purpose of the paper seems to be to point out that some of the disadvantages of the method of characteristics can be overcome by using another method based on replacing the differential equations by difference equations. *J. L. Ericksen.*

Glauz, R. D., and Lee, E. H. Transient wave analysis in a linear time-dependent material. J. Appl. Phys. **25**, 947–953 (1954).

It has become rather popular to use equations describing lumped systems of springs and dashpots as a guide in writing down constitutive equations for so-called viscoelastic materials. Using such a theory, the authors obtain a numerical solution for a semi-infinite bar, the end of which moves with constant velocity, assuming that the bar is initially unstressed, unstrained, and at rest. They point out that it is easy to obtain the solution for the case of constant stress applied to the end from their results.

J. L. Ericksen (Washington, D. C.).

Symonds, P. S., and Leth, C. F. A. Impact of finite beams of ductile metal. J. Mech. Phys. Solids **2**, 92–102 (1954).

The authors solve the problem of a uniform beam of arbitrary length, initially at rest, subject to a concentrated load applied at its midpoint in such a way as to cause the midpoint to move with constant velocity. Plastic rigid behavior is assumed. They also indicate how one can determine the motion ensuing if the load is subsequently removed.

J. L. Ericksen (Washington, D. C.).

Shield, R. T. Stress and velocity fields in soil mechanics. J. Math. Physics **33**, 144–156 (1954).

The paper is concerned with the two-dimensional stress and velocity fields for an ideal soil which yields when the stresses satisfy the well known Coulomb formula and then flows according to the stress-strain law formulated by Drucker and Prager [Quart. Appl. Math. **10**, 157–165

(1952); these Rev. 13, 1007]. After reviewing briefly the results of a previous paper where discontinuities in the velocity field were also considered [R. T. Shield, ibid. 11, 61–75 (1953); these Rev. 14, 931] the author investigates the discontinuities in the stresses. The results are applied to obtain stress and velocity fields for wedges loaded with uniform pressure on one side. For an acute-angled wedge the introduction of a line of stress discontinuity in the plastic stress region enables a stress solution to be found. The associated velocity field is then found in a similar manner to that used by Lee [Proc. Symposia Appl. Math., v. 3, McGraw-Hill, New York, 1950, pp. 213–228; these Rev. 12, 563] for the same problem in a Prandtl-Reuss material. It is also shown that the discontinuous stress solution for the obtuse-angled wedge can not be associated with a velocity field. A trapezoid loaded with the normal pressures on the parallel sides provides a further example of discontinuous stress fields. However the stress field obtained for this problem is physically inadmissible since a velocity field can not be associated with this stress field. *E. T. Onat.*

MATHEMATICAL PHYSICS

Rumer, Yu. B. The optical-mechanical analogy. Uspehi Matem. Nauk (N.S.) 8, no. 6(58), 55–69 (1953). (Russian)

This is a review article, tracing the development of the idea of an analogy between optics and dynamics, starting with Hamilton and ending with the author's 5-dimensional theory [Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. 23, 35–48 (1952); these Rev. 14, 706; and earlier papers there cited]. *F. J. Dyson* (Princeton, N. J.).

Optics, Electromagnetic Theory

***Montroll, E. W., and Greenberg, J. M. On the theory of scattering of plane waves by soft obstacles.** Proceedings of Symposia in Applied Mathematics, Vol. V, Wave motion and vibration theory, pp. 103–122. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1954. $7.00.

In this review paper the authors discuss various approximations which have been developed for the problem of scattering of waves by regions in which the index of refraction differs slightly from constant index correct for vacuo. The scattering region may be several wavelengths in size, and the index of refraction need not be constant so that the methods may be applied to quantum-mechanical as well as acoustical and electromagnetic problems. The authors first develop general approximate formulas which are applicable to spherical (cylindrical) regions of constant index of refraction. They then discuss a least-squares method which they have developed for replacing any spherically symmetrical scatterer satisfying the above conditions by an approximating sphere with constant index of refraction and can then use the solutions already developed for that case. Both differential and total cross-section formulas are given and applied to a variety of cases. *H. Feshbach.*

Moshinsky, Marcos. Description de la diffraction-hachage par une distribution de sources. C. R. Acad. Sci. Paris 238, 2395–2397 (1954).

Vajnshtejn, L. A. Propagation in semi-infinite waveguides. Six papers. Translated by J. Shmoys. Division of Electromagnetic Research, Institute of Mathematical Sciences, New York University, Research Rep. No. EM-63, iv+129 pp. (1954).

Translations of the following papers: Izvestiya Akad. Nauk SSSR. Ser. Fiz. 12, 144–165 (1948); ibid. 166–180 (1948); Akad. Nauk SSSR. Žurnal Tehn. Fiz. 18, 1543–1564 (1948); ibid. 19, 911–930 (1949); Doklady Akad. Nauk SSSR (N.S.) 74, 485–488 (1950); ibid. 909–912 (1950) [these Rev. 10, 659, 660; 12, 454, 462].

Pearson, J. D. A contribution to the theory of right-angled junctions in wave guides. Quart. J. Mech. Appl. Math. 7, 194–202 (1954).

The author examines the propagation of electromagnetic waves through an E-plane T-junction where a square waveguide leads into the broad face of a rectangular waveguide. He considers the case for which the rectangular waveguide is excited at its far end by an incident H_{10} mode and terminated at its near end by an electric wall. For various positions of the electric wall he computes the energy carried away from the junction by the E_{10}, E_{11}, and H_{11} modes of the unterminated square waveguide and by the E_{11} and H_{11} modes when the E_{10} mode is completely reflected by a filter placed at a sufficient distance from the junction. The author follows the Fourier-Lamé method of expanding the fields in mode functions and matching over the surfaces which bound the junction region. *C. H. Papas* (Pasadena, Calif.).

Poincelot, Paul. Sur la constante de temps d'un guide électrique cylindrique. C. R. Acad. Sci. Paris 238, 2394–2395 (1954).

***Gans, Ricardo, and Bemporad, Manuel. Contribution to the theory of the rectilinear antenna.** New research techniques in physics, pp. 427–444. Symposium organized by the Academia Brasileira de Ciências and Centro de Cooperación Científica para America Latina (UNESCO) under the auspices of the Conselho Nacional de Pesquisas do Brasil, Rio de Janeiro and São Paulo, July 15–29, 1952. Rio de Janeiro, 1954. (Spanish)

This article is the Spanish version of a paper originally published in German [Arch. Elektr. Übertragung 7, 169–180 (1953); these Rev. 15, 487]. Although the original has been reviewed, this reviewer believes that the article is still subject to some clarifying remarks based on quite different grounds.

The authors deduce an integral equation for the distribution $I(x)$ of the surface current along a perfectly conducting, cylindrical transmitting antenna of radius a and length $2l$ driven by a gap voltage $2V_0$. They convert the integral equation into an infinite system of simultaneous equations by expanding the current in a cosine series with undetermined Fourier coefficients and invoking the orthogonality property of the cosine functions [F. H. Murray, Amer. J. Math. 53, 873–890 (1931)]. They find that in the special case of resonance the cosine series converges rapidly and an accurate representation for $I(x)$ and for the input impedance is obtained by keeping only the first four terms. In spite of the fact that this method of orthogonal functions yields a valid result at resonance, one should not be led into believing that it will work as easily at, say, anti-resonance. Except at resonance the convergence problems become more than academic.

However, the main point that this reviewer wants to make concerns a case of mistaken identity of which the authors are guilty. The authors deduce early in their article the following integral equation (in slightly changed notation):

$$(1)\qquad \int_{-l}^{l} I(\xi)\frac{e^{-i\beta R}}{R}d\xi=\frac{4\pi}{Z_0}(V_0e^{-i\beta|x|}+A\cos\beta x),$$

where $Z_0=120\pi$ ohms, $\beta=2\pi/$free-space-wavelength, and $R=[a^2+(x-\xi)^2]^{1/2}$. They claim they have followed "Hallén's method" in the derivation of (1) and they ascribe (1) and an allegedly embarrassing property of (1), viz., that the x-derivative of its left side is a continuous function of x whereas the x-derivative of its right side is discontinuous, to Hallén. In fact, they go so far as to call (1) "Hallén's integral equation".

But the fact of the matter is that (1) is not Hallén's equation [Nova Acta Soc. Sci. Upsaliensis (4) 11, no. 4 (1938), p. 15]. And it has very little to do with either Hallén's method or with Hallén's linearized integral equation. To guess the source of error this reviewer recalls that the following equation does appear as an intermediate step in Hallén's method:

$$(2)\qquad \int_{-l}^{l} d\xi I(\xi)\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-i\beta r}}{r}d\varphi=\frac{4\pi}{Z_0}(V_0e^{-i\beta|x|}+A\cos\beta x),$$

where $r=[(x-\xi)^2+4a^2\sin^2\frac{1}{2}\varphi]^{1/2}$ and $\varphi=$azimuthal angle. Evidently the authors mistook (1) for (2). The difference, however, is indisputable and stems from the difference between R and r. Moreover, the celebrated linearized integral equation of Hallén is

$$(3)\qquad I(x)\log\frac{4(l^2-x^2)}{a^2}+\int_{-l}^{l}\frac{I(\xi)e^{-i\beta|x-\xi|}-I(x)}{|x-\xi|}d\xi=\frac{4\pi}{Z_0}(V_0e^{-i\beta|x|}+A\cos\beta x).$$

And this integral equation does not suffer from the difficulties that the authors complain of. *C. H. Papas.*

Gincburg, M. A. On the propagation of electromagnetic waves in a gyrotropic layer. Doklady Akad. Nauk SSSR (N.S.) **95**, 753–756; erratum **97**, 572 (1954). (Russian)

The author considers the propagation of a normally incident, linearly polarized electromagnetic wave through a homogeneous gyrotropic layer of thickness d. The medium to the left ($z\leqq 0$) and the medium to the right ($z\geqq d$) are homogeneous and isotropic. The gyrotropic layer ($0\leqq z\leqq d$) has a tensor dielectric constant and a tensor permeability. That is, $\epsilon=\epsilon^{(1)}$, $\mu=\mu^{(1)}$ when $z\leqq 0$; $\epsilon=\epsilon^{(2)}$, $\mu=\mu^{(2)}$ when $z\geqq d$; $\epsilon_{xx}=\epsilon_{yy}=\epsilon_1$, $\epsilon_{xz}=\epsilon_2$, $\epsilon_{yz}=-\epsilon_{xy}=i\epsilon_2$, $\mu_{xx}=\mu_{yy}=\mu_1$, $\mu_{xz}=\mu_2$, $\mu_{yz}=-\mu_{xy}=i\mu_2$ when $0\leqq z\leqq d$. By decomposing the waves into left and right circularly polarized components and enforcing the continuity of the electric and magnetic fields at the two surfaces, he obtains in a straightforward manner an expression for the transmitted wave. He computes the Faraday rotation as a function of the thickness d for the special case where the media to the left and to the right of the layer are the same. *C. H. Papas* (Pasadena, Calif.).

Durand, Émile. Sur les formules générales de la théorie des milieux polarisés. C. R. Acad. Sci. Paris **238**, 2504–2506 (1954).

Given a polarized volume $v(\xi,\eta,\zeta)$ containing neither charges nor currents and characterized by the density of electric polarization $\mathbf{P}(\xi,\eta,\zeta,t)$ and the density of magnetic polarization $\mathbf{M}(\xi,\eta,\zeta,t)$, two polarization potentials are defined at any point (x,y,z) by $\pi=(1/4\pi)\int_v[\mathbf{P}]dv/r$ and $\theta=(\mu_0/4\pi)\int_v[\mathbf{M}]dv/r$. The square brackets denote the functions at the retarded time $t-(r/c)$, and r is the distance between the points (x,y,z) and (ξ,η,ζ). It is shown that all electromagnetic formulas for polarized media can be formally derived from the expressions for π and θ. These formulas are generalizations of classical electromagnetic formulas. Different expressions are obtained in some cases depending on whether the point (x,y,z) is interior to v, is on the surface bounding v, or is exterior to that surface. (The article specifies that the point (x,y,z) is interior to v, is on the surface bounding v, or is interior to that surface. The latter statement appears to be in error.)

J. E. Rosenthal (Passaic, N. J.).

Kinokuniya, Yoshio. Some mathematical investigations on Doppler-effect. Mem. Muroran Univ. Engrg. **1**, 581–584 (1953).

Lorenzelli, Ezio. Acoustical, optical, or thermal phenomena induced on a plane by the presence of an arbitrary continuous surface which reflects a point source of sonic or electromagnetic energy. An. Soc. Ci. Argentina **156**, 69–87 (1953). (Spanish)

Dungey, J. W., and Loughhead, R. E. Twisted magnetic fields in conducting fluids. Australian J. Physics **7**, 5–13 (1954).

In this paper the propagation of waves in a twisted magnetic field $\mathbf{H}=(0,H_\varphi(r),H_z(r))$ (where r, φ, z denote cylindrical polar co-ordinates) is considered. Examining first the case when $H_\varphi=0$ and $H_z=$constant for $r<R$ and $\mathbf{H}=0$ for $r>R$, the author finds that the Alfvén velocity of propagation is reduced by a factor which tends to $(1+\rho_2/\rho_1)^{-\frac{1}{2}}$ for both long and short wave lengths. The case of a uniformly twisted field when $H_\varphi=Ar$ ($A=$constant) and $H_z=$constant for $r<R$ and $\mathbf{H}=0$ for $r>R$ is next considered in some detail. For solutions of the relevant equations which depend on t, φ and z in the manner $\exp i(\omega t+m\varphi+kz)$ the author establishes a characteristic equation for determining ω for given m and k. To examine the stability of the twisted field considered, ω is set equal to zero and the characteristic equation is reduced to the form

$$(*)\qquad F_m(\kappa R)=g_m(kp)$$

where $\kappa^2=k^2(4A^2/K^2-1)$, $K=kH_z+mp$, $p=H_z/A$, and

$$F_m(x)=xJ_m'(x)/J_m(x)\quad (J_m \text{ denotes the Bessel function})$$

and $g_m(kp)=4-(kp+m)^2-2m/(kp+m)$. It is shown that for solutions to exist one must have $(kp+m)^2<4$; and that the critical value of A for stability is given by $R=R_0$, where R_0 is the smallest value of R which satisfies (*). From a discussion of this last condition the author shows that the field first becomes unstable for modes which do not lead to the formation of loops. Nevertheless, the conclusion is reached that the "condition obtained for loop formation is that the pitch of the twisted field be less than πR".

S. Chandrasekhar (Williams Bay, Wis.).

Lehnert, B. Magnetohydrodynamic waves under the action of the Coriolis force. Astrophys. J. **119**, 647–654 (1954).

It is known that in a uniform magnetic field, $\mathbf{H}$, waves can be propagated in an incompressible fluid of infinite electrical conductivity with a velocity $V_s=H_z(\mu/\rho)^{1/2}$, where

H_s is the component of $\mathbf{H}$ in the direction of wave propagation (s), μ is the coefficient of absolute permeability, and ρ is the density. In this paper the author investigates the effect on these waves of a Coriolis acceleration $2\Omega\times\mathbf{v}$ resulting from a rotation with an angular velocity Ω and shows that each mode of wave propagation splits into two circularly polarized transverse waves with velocities given by $V_s[(1+\chi^2)^{1/2}\pm\chi]$, where $\chi=\Omega_s/(\kappa V_s)$ and κ is the wave number. *S. Chandrasekhar* (Williams Bay, Wis.).

Reulos, René. Sur l'intégration des équations de Maxwell et de Lorentz par la méthode opérationnelle. C. R. Acad. Sci. Paris **238**, 2225–2227 (1954).

In this note the author develops a slightly modified form of his "curl-series" solution of Maxwell's equations [same C. R. **208**, 423–425 (1939)]. *A. Erdélyi.*

Synge, J. L. On the transfer of energy between electromagnetic dipoles. Proc. Roy. Irish Acad. Sect. A. **56**, 1–11 (1954).

The author examines some of the properties of singularities of a classical oscillating electromagnetic field which act as sinks of energy. If such an "introvert" dipole is in the presence of an ordinary radiating dipole of the same strength, it is found that independently of their relative position there is no flow of energy to infinity. This is the electromagnetic analogue of a source and sink of equal strength in classical hydrodynamical theory.

H. C. Corben (Los Angeles, Calif.).

Huffman, D. A. The synthesis of sequential switching circuits. I, II. J. Franklin Inst. **257**, 161–190, 275–303 (1954).

A sequential switching circuit may be described as a relay circuit having a network of secondary relays whose state of operation may depend on a previous event in the primary relay circuit, i.e., the circuit connected directly to the control terminals. The transmission characteristics at the output terminals is a function of the states of both primary and secondary relay circuits. An important feature of the sequential circuit is the distinction between the excitation and response of a relay. This operating delay time is reflected in what the author terms the stability or instability of a relay. In effect, a relay is unstable during the transition state. Complications in sequential circuits arise when the various relays of the network have unequal transition times.

The analysis of sequential circuits is a procedure for enumerating the possible output terminal characteristics in terms of any of the possible input states. This is expedited by a transition matrix which accounts for the possible transition states of the secondary relay network. The synthesis procedure relies on a flow table which represents circuit conditions (states of primary and secondary relays) for which all secondary relays are stable. When, for a desired network realization, a flow table has been established the application of established procedures for the synthesis of combinational switching circuits (the primary and secondary circuits separately) leads to the desired result. The use of electronic circuits instead of relays is discussed.

R. Kahal (Monterey, Calif.).

Reza, F. M. Conversion of a Brune cycle with an ideal transformer into a cycle without an ideal transformer. J. Math. Physics **33**, 194–198 (1954).

The results obtained by Bott and Duffin [J. Appl. Phys. **20**, 816 (1949); these Rev. **12**, 307] for network driving-point impedance synthesis without transformers are recast into a stepwise procedure which may be termed a continuation of the Brune method [J. Math. Physics **10**, 191–236 (1931)]. It is a matter of opinion whether or not this constitutes an alternative proof to that of Bott and Duffin since the author's equations (2) and (5) together result in Bott-Duffin's generalization of a theorem by Richards.

R. Kahal (Monterey, Calif.).

Gilbert, E. N. Lattice theoretic properties of frontal switching functions. J. Math. Physics **33**, 57–67 (1954).

This paper is concerned with the kinds of switching operations which can be performed by switching circuits using either no negators or very few of them. The reason for this is that some types of switching circuit devices which can perform negation are more expensive or more unreliable than devices which perform other logical operations.

By an N-terminal contact network the author means a circuit built from relay contacts and interconnecting wires, and having N accessible terminals numbered from 1 to N but having no relay windings. Associated with an N-terminal contact network are two fundamental partially ordered systems, to be described shortly. If n is the number of relay magnets which control the contacts in the contact network, a relay state, x, is a sequence of binary digits $x=(x_1, \cdots, x_n)$, $x_i=0$ or 1 according as the ith relay magnet is turned off or on. If $x\leqq y$ is defined to mean $x_i\leqq y_i$ for $1\leqq i\leqq n$, then the set of relay states becomes a partially ordered system, in particular a lattice B^n. By a terminal state π is meant a partition of the numbers $1, 2, \cdots, N$ such that numbers belong to the same set in a partition if and only if they represent terminals which are connected together electrically by the contact network. By $\pi_1\leqq\pi_2$ is meant π_1 is a subpartition of π_2. Under this ordering the collection of terminal states becomes a lattice, $P(N)$. For a given contact network each relay state x determines a terminal state, π. The function which gives π in terms of x is called an N-terminal switching function. By a frontal network is meant a contact network in which all contacts are front contacts. A frontal switching function is the switching function of a frontal network. Theorem 1: N-terminal switching function f is frontal if and only if for all relay states x, y, $x\leqq y$ implies $f(x)\leqq f(y)$. Theorem 2: Suppose $P\subset P(N)$. A frontal network exists which produces all the terminal states of P and uses only n relays if and only if there is an order-preserving isomorphism between P and a subset of B^n. Numerical estimates are given concerning certain types of frontal switching functions.

S. Sherman (Philadelphia, Pa.).

Burks, Arthur W., and Wright, Jesse B. Theory of logical nets. Proc. I. R. E. **41**, 1357–1365 (1953).

The author proposes a theory of idealized electrical circuits. Each wire of these circuits is capable of being in two states, which are designated 0 and 1; these states change as functions of the variable t representing the time, which takes on the values 0, 1, 2, $\cdots$. There are two kinds of elementary circuits, which are called stroke elements and decision elements. A stroke element is a device consisting of two inputs and an output, such that the output is in state 0 whenever both inputs are in state 1 and the output is in state 1 if at least one of the inputs is in state 0. A delay element is a device with an input $f(t)$ and an output $g(t)$, such that $g(0)=0$ and $g(t+1)=f(t)$. A logical net is an array formed by interconnecting the inputs and outputs of a finite number of these elements. It is shown that it is possible

to construct logical nets having freakish characteristics such as could not be realized by actually useful circuits. The author defines various special kinds of nets (well-behaved, deterministic, well formed) which exclude these freak possibilities. He studies the relationships of these different kinds to one another and to other criteria of realizability.

H. B. Curry (State College, Pa.).

Quantum Mechanics

Kompaneec, A. S. The equations of the self-consistent field for a nucleus taking account of electrostatic forces. Akad. Nauk SSSR. Žurnal Eksper. Teoret. Fiz. **26**, 153–158 (1954). (Russian)

An approximation, analogous to that of Thomas-Fermi, is applied to a distribution of protons and neutrons in scalar interaction to obtain a rather complicated set of equations which may be solved by the self-consistent field method. These equations involve two distribution functions for the proton and neutron fields and two potential functions. The potentials are determined as solutions of two second-order differential equations involving the distribution functions. The present paper completes the author's previous work [Doklady Akad. Nauk (N.S.) **85**, 301–304 (1952)] by formulating precise boundary conditions for the solutions of these differential equations. The test of consistency of an assumed distribution with the resulting potential is provided by two equations which contain the logarithm and inverse tangent of the distributions.

A. J. Coleman (Toronto, Ont.).

Miller, S. C., Jr. Normalization of WKB-type approximations. Physical Rev. (2) **94**, 1345–1346 (1954).

A method is given for normalizing WKB-type approximations to the wave functions of a simple potential well. The method is similar to one given by W. H. Furry [Physical Rev. (2) **71**, 360–371 (1947); these Rev. **8**, 463] for finding the normalization of the usual WKB approximations. The normalization constants are given in terms of those of a qualitatively similar, but solvable, problem. The author states that he cannot estimate the error in the approximation or generalize the method to give matrix elements.

T. E. Hull (Vancouver, B. C.).

Weizel, Walter. Ableitung der quantenmechanischen Wellengleichung des Mehrteilchensystems aus einem klassischen Modell. Z. Physik **136**, 582–604 (1954).

The Schroedinger equation is given an interpretation in terms of classical particles whose motion is disturbed by a continuous isotropic shower of "zerons" which, on the average, contribute neither energy nor momentum to the particles. Somewhat along the lines of Bohm's procedure, two equations are obtained for the modulus and phase of the quantum-mechanical wave function. The two equations are interpreted as statements of the conservation of the number and the momentum of the particles. Conversely, from these two real equations together with the added assumption that the velocity vector of the particles is rotationless, it is possible to derive the Schroedinger equation.

A. J. Coleman (Toronto, Ont.).

Schwinger, Julian. The theory of quantized fields. VI. Physical Rev. (2) **94**, 1362–1384 (1954).

This paper deals with the theory of the Dirac electron-positron field in interaction with an unquantized and time-independent electromagnetic field. The formal theory is worked out following the methods of the author's earlier papers [Physical Rev. (2) **93**, 615–628 (1954); these Rev. **15**, 586; and other papers there cited]. The results of the analysis consist mainly of equations relating the energies of bound states of electrons, the phase-shifts of continuum states, and the various Green's functions of the Dirac field. An elegant result, typical of many others, is the following formula for the energy of the vacuum state of the Dirac field:

$$E(0)=\tfrac{1}{2}\sum_k(m-E_k)+\frac{1}{2\pi}\int_m^\infty dE\sum_\gamma \delta_{\gamma E}.$$

Here the first sum extends over the discrete states k with energies E_k, the second term is summed and integrated over the continuum states specified by their energy E and other quantum numbers γ, and $\delta_{\gamma E}$ is the phase shift for such a continuum state. Finally, m is the electron mass.

F. J. Dyson (Princeton, N. J.).

Edwards, S. F., and Peierls, R. E. Field equations in functional form. Proc. Roy. Soc. London. Ser. A. **224**, 24–33 (1954).

This important paper (under a rather misleading title) starts a new development in field theory. Schwinger's functional equation for a nucleon Green's function is solved in a linear approximation. The result is expressed as a functional integral, involving the nucleon Green's function in an arbitrary "external mesic" field. The method is applied to the case of neutral scalar theory, nucleon recoil being neglected. The closed form of the solution enables a thorough discussion of the singularities of the Green's function.

A. Salam (Cambridge, England).

Claesson, Arne. On the renormalization of the Salpeter-Bethe equation. Ark. Fys. **7**, 565–585 (1954).

By a detailed calculation, the author verifies that all divergences in Salpeter-Bethe equation can be compensated up to the fourth order, if renormalization terms are introduced into the Lagrangian [see G. Källén, Helvetica Phys. Acta **25**, 417–434 (1952); these Rev. **14**, 435].

A. Salam (Cambridge, England).

Matthews, P. T., and Salam, Abdus. Renormalization. Physical Rev. (2) **94**, 185–191 (1954).

This is a review article, summarizing in a brief and readable form the logical basis of the renormalization method in quantum field theory. Working with fully renormalized field operators, a method introduced independently by G. Takeda [Progress Theoret. Physics **7**, 359–366 (1952)] and by G. Källén [Helvetica Phys. Acta **25**, 417–434 (1952); these Rev. **14**, 435], the authors obtain a remarkably simple description of the renormalization process. *F. J. Dyson.*

Kar, S. C. Versuch einer logischen Quantendynamik des Elektrons. II. Bull. Calcutta Math. Soc. **45**, 133–171 (1953).

This paper consists of applications of the author's theory of the electron [same Bull. **44**, 1–21 (1952); these Rev. **14**, 607] to the following standard problems: motion of a free electron, motion of an electron in a constant electric or magnetic field, motion of an electron in a Coulomb field. The equations of motion are solved in detail for each case, and it is verified that the solutions reduce to the classical orbits in the limit as Planck's constant h tends to zero. The quantum solutions are, however, entirely different from those of orthodox quantum mechanics. For example, the

energy levels in a Coulomb field are given by

$$(1)\qquad E=mc^2\left[1-\frac{\alpha^2Z^2}{(s-1+[(l+1)^2+\alpha^2Z^2]^{1/2})^2}\right]^{-1/2},$$

where α is the fine-structure constant, and s and l are integers. The Dirac theory gives instead of (1)

$$(2)\qquad E=mc^2\left[1+\frac{\alpha^2Z^2}{(s-1+[(l+1)^2-\alpha^2Z^2]^{1/2})^2}\right]^{-1/2},$$

a result which is well confirmed by experiment. The energy levels (1) all lie above mc^2, so that their physical meaning is not clear. *F. J. Dyson* (Princeton, N. J.).

Havas, Peter. On the classical theory of particles interacting with electromagnetic and mesonic fields. I. Physical Rev. (2) **93**, 882–888 (1954).

The classical equations of motion are obtained for particles in an electromagnetic and in a neutral scalar or vector meson field, each equation being written for (a) retarded potentials (b) half advanced, half retarded potentials, (c) the theory of action at a distance. The relations between the experimental and mechanical masses are exhibited for the various theories. Cross-sections are evaluated for the scattering of electromagnetic radiation, photo-meson and meso-photon production, and the scattering of mesons due to electromagnetic effects. *H. C. Corben.*

Havas, Peter. Multipole singularities of classical scalar and pseudoscalar meson fields. Physical Rev. (2) **93**, 1400–1411 (1954).

Following the methods of Dirac [Proc. Roy. Soc. London. Ser. A. **167**, 148–169 (1938)], Harish-Chandra [Proc. Roy. Soc. A. **185**, 269–287 (1946); these Rev. **7**, 538] succeeded in obtaining the equations of motion of a particle possessing multi-pole singularities, and which interacts with a neutral scalar or pseudoscalar field. In this paper, the author imposes the restrictions that the spin and the 2^n-pole moments of the particle are of constant magnitude, and have only spatial components in the rest system of the particle. Notwithstanding these, the general form of the multipole moments and the equations of motion can be determined, the formalism being extended to the case of particles interacting with charge-symmetric meson fields. *A. Salam.*

Morel-Viard. Les opérateurs α-correspondants en mécanique ondulatoire. C. R. Acad. Sci. Paris **238**, 992–994 (1954).

Werle, J. A new approximation method for the meson field theory. Bull. Acad. Polon. Sci. Cl. III. **1**, 23–26 (1953).

The method of Einstein, Infeld and Hoffmann [Ann. of Math. (2) **39**, 65–100 (1938)] is applied to the equations of meson field theory. It is assumed that the solution is quasi-static, so that it is taken as an infinite series, according to the orders of time derivatives: differentiating a term increases its order by one. By equating terms of the same order one obtains an infinite series of simultaneous equations which can be solved in succession by the use of Fourier integrals. In this way one can obtain a series containing only even orders which, when the operator characterizing the field sources contains no time differentiation, is symmetrical with respect to change of time direction and corresponds to half the sum of the retarded and advanced potentials. The series with odd orders corresponding to half the difference of these potentials goes to infinity at infinity. The author also obtains an expression for the solution in terms of an integral operator. *N. Rosen* (Haifa).

Chraplyvy, Zeno V. Reduction of relativistic two-particle wave equations to approximate forms. II. Physical Rev. (2) **92**, 1310–1315 (1953).

In a previous work the author proposed a procedure for transforming the Dirac Hamiltonian for a two-particle system so as to separate the components of the Dirac wave function for positive and negative energy states [Physical Rev. (2) **91**, 388–391 (1953); these Rev. **15**, 382]. The method failed in the case of two particles of equal masses. In the present paper the author imposes less stringent conditions on the matrices in the transformed Hamiltonian than in the earlier paper. For example, instead of requiring that the matrices have no elements between any two states with different signs of particle energies, he requires only that they have no elements between the states in which both particles have positive energy and the other states. In this way he obtains an approximate form of the two-particle wave equation which is meaningful also for the case of equal masses. *N. Rosen* (Haifa).

Hughes, J. B. Note on a paper by K. H. Tzou. Philos. Mag. (7) **44**, 1300–1302 (1953).

K. H. Tzou [Philos. Mag. (7) **39**, 790–799 (1948); these Rev. **10**, 417] discussed the motion of a free electron using relativistic Heisenberg equations; he stated that the results have some deviation from those obtained by Dirac using non-relativistic Heisenberg equations. J. B. Hughes shows in this note that the results are, in fact, equivalent.

A. E. Schild (Pittsburgh, Pa.).

Costa de Beauregard, Olivier. Théorie relativiste covariante de la particule liée. C. R. Acad. Sci. Paris **238**, 1196–1198 (1954).

BIBLIOGRAPHICAL NOTES

***Chaundy, T. W., Barrett, P. R., and Batey, Charles. The printing of mathematics. Aids for authors and editors and rules for compositors and readers at the University Press, Oxford.** Oxford University Press, London, 1954. ix+105 pp. (4 plates). 15s. 7d.

The first part of this book, by Batey, discusses the mechanics of mathematical printing. The second part, by Chaundy and Barrett, contains recommendations to mathematical authors. The third part gives rules for composition of mathematics at the University Press, Oxford. Three appendices give recommended forms for handwritten letters, lists of special fonts and symbols, and lists of abbreviations.

***Iwanami sûgaku jiten. Nippon Sûgaku Kwai henshu. [The Iwanami mathematical dictionary. Compiled by the Japanese Mathematical Society.]** Iwanami Shoten, Tokyo, 1954. xx+591+98+89 pp. (10 plates). 1200 Yen.

This remarkable dictionary was published under the general editorship of Shokiti Iyanaga with the cooperation

of some 190 Japanese mathematicians. It begins with a list of symbols and notations, a list of abbreviations of journals, and a subject index. The next 591 pages are devoted to the dictionary proper. Mathematical terms are listed alphabetically according to a romanized spelling of Japanese. The Japanese characters for the term and the English, French, and German equivalents follow. The term is then defined and references to original papers (even as recent as 1953) and to sources for further study are given. The discussions are brief but not so perfunctory as to say nothing; for example, "differential ring" and "Kähler variety" each take about one column, "fibre bundle" takes eight, and "mathematics of heredity" takes three. Biographies of twenty-nine mathematicians are included.

Two appendices follow. The first contains lists of formulas (e.g., integrals, special series, special functions, special conformal mappings, special transforms), a chart showing how the simple Lie groups are related, a table of homotopy groups $\pi_{r+k}(S^r)$ for $r=1$ to 9 and $k=1$ to 8, tables of many useful special functions, and finally bibliographical information concerning the most important mathematical journals.

The second appendix contains two indices to the various terms used in the dictionary, not only to the main heading of an article, but also to further terms introduced in the course of discussing a topic; the first index is to the Japanese terms, written in Japanese characters, the second index is to terms in English, French, or German. This appendix concludes with a list of names of the mathematicians referred to in the dictionary. Where known, the dates of birth and death (if applicable) are given.

*Sûgaku no gaikan (1940–1949). Nippon Sûgaku Kwai hen. Nippon Gakujitsu Shinkô Kwai hakko. [Survey of mathematics (1940–1949). Compiled by the Japanese Mathematical Society. Published by the Japanese Society for the Advancement of Culture.] Tokyo, 1951. v+187 pp. 250 Yen.

This book, edited by Y. Komatu, contains a list of the mathematical papers published in Japanese journals during the years 1940–1949. A short summary in Japanese follows many of the listed papers. They are arranged according to the following fields: Foundations; algebra and number theory; geometry; function theory; topology; functional analysis; differential and integral equations; theory of real functions; probability theory; mathematical statistics; applied mathematics. The usefulness of the book for non-Japanese would have been increased if the romanized spelling of Japanese names had been included (say, in parentheses) alongside the Japanese characters for the name, and if the journals in the initial list had also been identified by their corresponding titles in a western language when such exists or by romanized spelling otherwise.

Acta Scientia Sinica.

Vol. 1, no. 1, is dated October 1952. This journal replaces the Science Record of the Academia Sinica which terminated with vol. 5 Acta. Scientia Sinica may be purchased through Guozi Shudian, 38 Suchou Hutung, Peking, China.

Annales Polonici Mathematici.

Vol. 1, no. 1, appeared in June 1954. This is a publication of the Polish Academy of Sciences (Polska Akademia Nauk) and replaces the Annales de la Société Polonaise de Mathématique which terminated with vol. 25. Subscriptions may be entered with Książka i Prasa, Koszykowa 31, Warsaw, Poland.

Bŭlgarska Akademiya na Naukite. Izvestiya na Matematičeskiya Institut.

Vol. I, no. 1 (Tom I, kniga pŭrva) is dated 1953. Papers are in Bulgarian with summaries in Russian.

Journal of the Association for Computing Machinery.

Vol. 1, no. 1, is dated January 1954. The journal is to appear quarterly and is published by the Association for Computing Machinery, 2 East 63rd St., New York 21, N. Y.

Journal of the Society for Industrial and Applied Mathematics.

Vol. 1, no. 1, is dated September 1953. Subscriptions may be placed with I. E. Block, Burroughs Corp., 1209 Vine St., Philadelphia 7, Pa.

Mathematika.

Vol. 1, part 1, is dated June 1954. Mathematika is published by the Department of Mathematics, University College, London. Subscriptions may be addressed to: Mathematika, University College, Gower Street, London, W.C.1, England.

Referativnyĭ Žurnal.

The first issues of this new reviewing journal appeared in October 1953. It began with five sections: Matematika, Mehanika, Astronomiya, Fizika, and Himiya. Other sections will be added later. The first four sections appear monthly. The journal is a publication of the Institut Naučnoĭ Informacii (Institute of Scientific Information) of the Akademiya Nauk SSSR. In the section Matematika borderline fields of mathematics are not as extensively covered as in Mathematical Reviews (or the Zentralblatt für Mathematik), since these papers are covered in the other sections. However, a paper on, say, elasticity theory may appear, with different reviews, both in the section Mehanika and in the section Matematika under the heading "Applications of differential equations." Within the field of mathematics proper coverage is considerably more extensive than in Mathematical Reviews. This is partly a result of coverage of publications which have not been available outside the USSR and partly a result of a decision in principle to broaden the coverage (e.g., reports of meetings, book reviews, patents). The coverage of computing machinery is notable. The completeness of the bibliographical data is also notable: Non-Russian titles are translated into Russian, the author's name is transcribed approximately phonetically into Russian, and then the title and author's name are given in the original language without transcription (even if it be Chinese). The rest of the data are given approximately as in Mathematical Reviews. All reviews in a given section are numbered. All reviews are in Russian. Reviewers are either from the USSR or bordering countries. The quality of reviewing is high and the printing and paper are excellent. Subscriptions may be entered for the individual sections.